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A Collocation Technique Based on Orthogonal Polynomials for Construction of Continuous Hybrid Methods.

Adeniyi, R. B.*[1] and Yahaya, H.[2]
[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2]Department of Mathematics, Sa'datu Rimi College of Education, Kumbotso –Kano, Nigeria.

Abstract

The paper focuses on formulation of a number of algorithms for the numerical solution of first order ordinary differential equations with applications to initial value problems. For this purpose, an orthogonal polynomial valid in interval [-1,1] and with respect to weight function $w(x)=x^2$ was constructed and employed as basis function for the development of some continuous hybrid schemes in a collocation and interpolation technique. To make the continuous schemes self-starting, some block methods of discrete hybrid form were derived. The schemes were analyzed using appropriate existing theorems to investigate their stability, consistency and convergence. The investigation shows that the developed schemes are consistent, zero-stable and hence convergent. The self-starting methods were implemented on some test problems from the literature to show the accuracy and effectiveness of the schemes.

Keyword: Interpolation, collocation, continuous, hybrid, schemes, orthogonal polynomials, convergence,

1. Introduction

The desirability of deriving a continuous scheme for solving first order and higher order ordinary differential equations cannot be over emphasized. This is as a result of the need to increase the effectiveness and efficiency of multistep methods in solving differential equations. Over the years, techniques for the derivation of Linear Multistep Methods (LMMs) for the numerical solution of the Initial Values Problems (IVPs) in first order ordinary differential equation of the form:

$$y' = f\big(x, y(x)\big), \ \ a \le x \le b < +\infty \qquad (1.1a)$$

$$y(a) = y_0 \qquad (1.1b)$$

have been reported in the literature, Onumanyi et al (1993) and Adeniyi et al (2006, 2007, 2008). These include, among others, collocation, interpolation, integration of interpolation polynomials.

*Corresponding Author: Adeniyi, R. B.
Email: raphade@unilorin.edu.ng

Differential equations play an important role in the modeling of physical problems arising from almost every discipline of study. However, it is relatively uncommon for a differential equation to have a solution that can be written in terms of elementary functions. Usually the only information about the solution is that it is known to exist and to be unique. Numerous ordinary differential equation solvers have been developed and implemented since the digital computer was introduced some decades ago. The criteria that are normally put into consideration are whether or not the solver gives an accurate approximation, uses less computation time, is easy to implement and obtains a unique numerical solution.

Most conventional ODE solvers such as Runge-Kutta, Taylor's algorithm and Linear Multistep Methods (LMMs) are easy to implement and also meet the first two criteria in normal circumstances. Among these methods, LMMs are very popular and useful. They are very suitable in providing solutions to ODEs within a given interval and they are also useful for information about the solution at more than one point. However, the effectiveness of these ODE solvers depends on the types of trial functions used in developing the schemes. Various trial functions such as, the Chebyshev polynomials $T_n(x)$, the Legendre polynomials $P_n(x)$, the monomials x^r, and the Canonical polynomials $(Q_r(x), r \geq 0)$of the Lanczos Tau method in a perturbed collocation approach have been employed for this purpose, Henrici (1962), Fox and Parker(1968), Lambert (1973), Lanczos (1973). Problems arising from ODEs can either be formulated as an Initial Value Problems (IVPs) or a Boundary Value Problems (BVPs).

In the quest for solution to general second order ODEs, Anake (2011) derived finite difference methods by power series in the form:

$$y(x) = \sum_{j=0}^{k} a_j x^j \qquad (1.2)$$

for the solution of IVPs for ODEs. This proposed power series based one-step hybrid method was developed by appropriate selection of points for both interpolation and collocation to obtain block methods through which many useful classes of finite difference methods were generated to implement the new method. Adeyefa (2014) employed the polynomial

$$T_n(x) = \cos\left[ncos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right] = \sum_{r=0}^{n} C_r^{(n)} x^r \quad (1.3)$$

to develop many classes of finite difference methods for the solution of second order IVPs in ODEs.

In this work, our concern is on the first order IVPs and the choice of the orthogonal polynomials shall be considered to develop a class of continuous hybrid block methods which simultaneously generates solutions of (1) without the need for any predictor.

2.0 Derivation of the Methods

We consider the derivation of a class of continuous schemes which shall be used to generate the discrete counterparts required to set up the block methods. This we do by approximating the analytical solution of problem (1.1) with the orthogonal polynomials which shall be defined and constructed hereunder.

2.1 The Orthogonal Polynomial

Consider the equation

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)dx = h_n\delta_{mn} \quad (2.1)$$

with

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

where the weight function w(x) is continuous and positive on [a, b] such that the moments

$$\mu = \int_a^b w(x)x^n dx, \quad n = 0,1,2, \dots\dots\dots \quad (2.2)$$

exist and h_n is a non-zero constant.

Then the integral

$$< \varphi_m, \varphi_n >= \int_a^b w(x)\varphi_m(x)\varphi_n(x)dx \quad (2.3)$$

denotes an inner product of the polynomials φ_m and φ_n.

For orthogonality,

$$< \varphi_m, \varphi_n >= \int_a^b w(x)\varphi_m(x)\varphi_n(x)dx = 0, \; m \neq n, \quad [-1,1] \quad (2.4)$$

In this study, we shall adopt the weight function $w(x) = x^2$ in the [a,b]≡[-1,1].

The construction of the basis function $\varphi_n(x), n$= 1, 2, 3 ... of the approximant:

$$y_n(x) = \sum_{r=0}^{n} a_r \varphi_r(x) \cong y(x) \quad (2.5)$$

now follows:

2.1.1 Construction of Orthogonal Basis Function

For the purpose of constructing the basis function, we use additional property that

$$\varphi_n(1) = 1$$

where $\varphi_n(x)$ defined by

$$\varphi_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \quad (2.6)$$

satisfies the orthogonality property (2.4), i.e.,

$$< \varphi_m, \varphi_n > \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

Concisely, we have that

$$\varphi_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r$$

$$<\varphi_m, \varphi_n> = 0, \; m \neq n$$

$$\varphi_n(1) = 1.$$

When r = 0 in (2.6)

$$\varphi_0(x) = C_0^{(0)}.$$

By definition

$$\varphi_0(1) = C_0^{(0)} = 1.$$

Hence

$$\varphi_0(x) = 1.$$

The other orthogonal polynomials $\varphi_r(x),\; 1 \leq r \leq 7$ are developed in the same way and employed in this work. The first few of them are listed below:

$$\varphi_1(x) = x,$$

$$\varphi_2(x) = \frac{1}{2}(5x^2 - 3),$$

$$\varphi_3(x) = \frac{1}{2}(7x^3 - 5x),$$

$$\varphi_4(x) = \frac{1}{8}(63x^4 - 70x^2 + 15),$$

$$\varphi_5(x) = \frac{1}{8}(99x^5 - 126x^3 + 35x),$$

$$\varphi_6(x) = \frac{1}{16}(429x^6 - 693x^4 + 315x^2 - 35),$$

$$\varphi_7(x) = \frac{1}{16}(715x^7 - 1287x^5 + 315x^3 - 105x).$$

In what now follows, we shall use the derived polynomials to develop the proposed continuous schemes.

We set out by considering the IVP (1.1) in the subinterval $[x_n, x_{n+p}]$ of

$y' = f\big(x, y(x)\big),\ \ x_n \le x \le x_{n+p}$, where p = 1, 2, 3, ... (2.7)

We seek an approximation of the form

$$Y(x) = \sum_{r=0}^{k-1} a_r \varphi_r(x) \cong y(x), \quad x_n \le x \le x_{n+p}$$

where p varies as the method to be derived and $n = 0,1,2,$ .... That is

$$Y(x) = \sum_{r=0}^{k} a_r \varphi_r \left(\frac{2(x - x_n)}{ph} - 1\right) \quad , x_n \le x \le x_{n+p} \qquad (2.8)$$

where $h = x_{n+1} - x_n$ is the uniform step-length.

The procedures involve interpolating (2.8) at both grid and off-grid points while we collocate the first order derivative of (2.8) at some grid and off-grid points. The coefficients of the resulting system of equations will thereafter be determined through the Gaussian elimination method and, the values of which will be substituted into (2.8). Consequently, we construct a continuous implicit hybrid method. The continuous implicit hybrid method will be evaluated at the collocation points to yield some corresponding discrete hybrid schemes which constitute a block through which the solution of (1.1) will be obtained. In what now immediately follows, we shall develop one-step, two-step and three-step hybrid methods.

2.2 Derivation of Continuous One-Step Method with One Off-step Point

In this section, we derived a continuous one step method with one off-step point.

Considering equation (2.8) for k = 4 and p = 1, we have

$$Y(x) = \sum_{r=0}^{3} a_r \varphi_r \left(\frac{2(x - x_n)}{h} - 1\right) \qquad (2.9)$$

where $x \in [x_n, x_{n+1}]$.

Expanding (2.9), we obtain

$$Y(x) = a_0 + a_1\left(\frac{2(x-x_n)}{h} - 1\right) + a_2\left(\frac{5}{2}\left(\frac{2(x-x_n)}{h} - 1\right)^2 - \frac{3}{2}\right) + a_3\left(\frac{7}{2}\left(\frac{2(x-x_n)}{h} - 1\right)^3 - \frac{5}{2}\left(\frac{2(x-x_n)}{h} - 1\right)\right) \qquad (2.10a)$$

by engaging the first four orthogonal polynomials obtained earlier. Expanding (2.10a) and simplifying in terms of a function t, we have

$$Y(t) = a_0 + a_1(2t-1) + a_2(10t^2 - 10t + 1) + a_3(28t^3 - 42t^2 + 16t - 1) \qquad (2.10b)$$

where $t = \frac{x-x_n}{h}$.

Differentiating (2.10b), we obtain

$$Y'(t) = 2a_1 + a_2(20t - 10) + a_3(84t^2 - 84t + 16) \qquad (2.11)$$

Then, collocating (2.11) at t = 0, $\frac{1}{2}$, 1 and interpolating (2.10b) at t=0 lead to a system of equations written in the matrix form AX = B as:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -10 & 16 \\ 0 & 2 & 0 & -5 \\ 0 & 2 & 10 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{2}} \\ hf_{n+1} \end{bmatrix} . \qquad (2.12)$$

Equation (2.12) is solved by Gaussian elimination method to obtain the value of the unknown parameters a_j, j = 0, 1, 2, 3 as follows:

$$a_0 = y_n + \frac{h}{30}\left(f_{n+1} + 10f_{n+\frac{1}{2}} + 4f_n\right),$$

$$a_1 = \frac{h}{84}\left(f_{n+1} + 32f_{n+\frac{1}{2}} + 5f_n\right), \qquad (2.13)$$

$$a_2 = \frac{h}{20}(f_{n+1} - f_n),$$

$$a_3 = \frac{h}{42}\left(f_{n+1} - 2f_{n+\frac{1}{2}} + f_n\right).$$

Substituting (2.13) into (2.10) yields a continuous implicit hybrid one-step method in the form

$$y(x) = \alpha_j(x)y_n + h\left(\sum_{j=0}^{1} \beta_j(x)f_{n+j} + \beta_j(x)f_{n+\frac{1}{2}}\right) , \qquad (2.14)$$

where $\alpha_j(x)$and $\beta_j(x)$are continuous coefficients obtained as:

$$\alpha_0(t) = 1,$$

$$\beta_0(t) = \frac{2}{3}t^3 - \frac{3}{2}t^2 + t,$$

$$\beta_{\frac{1}{2}}(t) = \frac{-4}{3}t^3 + 2t^2,$$

$$\beta_1(t) = \frac{2}{3}t^3 - \frac{1}{2}t^2.$$

Evaluating (2.14) at x_{n+1} and $x_{n+\frac{1}{2}}$ gives the discrete schemes

$$y_{n+1} = y_n + \frac{h}{6}\left(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}\right), \qquad (2.15)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{24}\left(5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}\right). \qquad (2.16)$$

2.3 Derivation of Continuous Two-Step Method with Two Off-Step Points

We consider here in this section the derivation of continuous two-step method with two off-step points. In equation (2.8), if k = 6 and p = 2, we will obtain

$$Y(x) = \sum_{r=0}^{3} a_r \varphi_r(x) \quad . \qquad (2.17)$$

Expanding (3.17), we have

$$Y(x) = a_0 + a_1\left(\frac{(x-x_n}{h} - 1\right) + a_2\left(\frac{5}{2}\left(\frac{(x-x_n)}{h} - 1\right)^2 - \frac{3}{2}\right) + a_3\left(\frac{7}{2}\left(\frac{(x-x_n)}{h} - 1\right)^3 - \frac{5}{2}\left(\frac{(x-x_n)}{h} - 1\right)\right) + a_4\left(\frac{63}{8}\left(\frac{(x-x_n)}{h} - 1\right)^4 - \frac{70}{8}\left(\frac{(x-x_n}{h} - 1\right)^2 + \frac{15}{8}\right) + a_5\left(\frac{99}{8}\left(\frac{(x-x_n)}{h} - 1\right)^5 - \frac{126}{8}\left(\frac{(x-x_n}{h} - 1\right)^3 + \frac{35}{8}\left(\frac{(x-x_n}{h} - 1\right)\right). \qquad (2.18)$$

For simplicity, equation (2.18) is arranged as

$$Y(t) = a_0 + a_1(t-1) + a_2\left(\frac{5}{2}(t-1)^2 - \frac{3}{2}\right) + a_3\left(\frac{7}{2}(t-1)^3 - \frac{5}{2}(t-1)\right) + a_4\left(\frac{63}{8}(t-1)^4 - \frac{70}{8}(t-1)^2 + \frac{15}{8}\right) + a_5\left(\frac{99}{8}(t-1)^5 - \frac{126}{8}(t-1)^3 + \frac{35}{8}(t-1)\right), \qquad (2.19)$$

where $t = \frac{x-x_n}{h}$.

Differentiating (2.19), we obtain

$$Y'(t) = a_1 + a_2(5t-5) + a_3\left(\tfrac{21}{2}t^2 - 21t + 8\right) + a_4\left(\tfrac{63}{2}t^3 - \tfrac{189}{2}t^2 + 77t - 14\right) + a_5\left(\tfrac{495}{8}t^4 - \tfrac{495}{2}t^3 + 324t^2 - 153t + 19\right). \quad (2.20)$$

Interpolating (2.19) at t = 0 and collocating (2.20) at t = 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2, we have a system of equations written in the matrix form AX = B as

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -5 & 8 & -14 & 19 \\ 0 & 1 & \frac{-5}{2} & \frac{1}{8} & \frac{77}{16} & \frac{-457}{128} \\ 0 & 1 & 0 & \frac{-5}{2} & 0 & \frac{35}{8} \\ 0 & 1 & \frac{5}{2} & \frac{1}{8} & \frac{-77}{16} & \frac{-457}{128} \\ 0 & 1 & 5 & 8 & 14 & 19 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ hf_{n+\frac{3}{2}} \\ hf_{n+2} \end{bmatrix}. \quad (2.21)$$

Solving equation (2.21) using Gaussian elimination method, we obtain the value of the unknown parameters a_j, j= 0(1)5 as follows:

$$\begin{aligned} a_0 &= y_n + \frac{h}{315}\left(44f_n + 136f_{n+\frac{1}{2}} + 42f_{n+1} + 88f_{n+\frac{3}{2}} + 5f_{n+2}\right), \\ a_1 &= \frac{h}{378}\left(13f_n + 128f_{n+\frac{1}{2}} + 96f_{n+1} + 128f_{n+\frac{3}{2}} + 13f_{n+2}\right), \\ a_2 &= \frac{h}{270}\left(-11f_n - 32f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{2}} + 11f_{n+2}\right), \quad (2.22) \\ a_3 &= \frac{h}{3465}\left(113f_n + 208f_{n+\frac{1}{2}} - 642f_{n+1} + 208f_{n+\frac{3}{2}} + 113f_{n+2}\right), \\ a_4 &= \frac{h}{189}\left(-4f_n + 8f_{n+\frac{1}{2}} - 8f_{n+\frac{3}{2}} + 4f_{n+2}\right), \\ a_5 &= \frac{h}{1485}\left(16f_n - 64f_{n+\frac{1}{2}} + 96f_{n+1} - 64f_{n+\frac{3}{2}} + 16f_{n+2}\right). \end{aligned}$$

Substituting (2.22) into (2.19) yields a continuous implicit hybrid two-step method in the form

$$y(t) = \alpha_j(t)y_n + h\sum_{j=0}^{2}\beta_j(t)f_{n+j} + \beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}f_{n+\frac{3}{2}}, \quad (2.23)$$

where $\alpha_j(x)$and $\beta_j(x)$are continuous coefficients obtained as:

$$\alpha_0(t) = 1,$$

$$\beta_0(t) = \frac{2}{15}t^5 - \frac{5}{6}t^4 + \frac{35}{18}t^3 - \frac{25}{12}t^2 + t,$$

$$\beta_{\frac{1}{2}}(t) = \frac{-8}{15}t^5 + 3t^4 - \frac{52}{9}t^3 + 4t^2,$$

$$\beta_1(t) = \frac{4}{5}t^5 - 4t^4 + \frac{19}{3}t^3 - 3t^2,$$

$$\beta_{\frac{3}{2}}(t) = \frac{-8}{15}t^5 + \frac{7}{3}t^4 - \frac{28}{9}t^3 + \frac{4}{3}t^2,$$

$$\beta_2(t) = \frac{2}{15}t^5 - \frac{1}{2}t^4 + \frac{11}{18}t^3 - \frac{1}{4}t^2.$$

Evaluating (2.23) at $x_{n+\frac{1}{2}}$, x_{n+1}, $x_{n+\frac{3}{2}}$ and x_{n+2} the following discrete schemes are obtained:

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{1440}\left(251f_n + 646f_{n+\frac{1}{2}} + 264f_{n+1} + 106f_{n+\frac{3}{2}} + 19f_{n+2}\right),$$

$$y_{n+1} = y_n + \frac{h}{180}\left(29f_n + 124f_{n+\frac{1}{2}} + 24f_{n+1} + 4f_{n+\frac{3}{2}} - f_{n+2}\right), \qquad (2.24)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{160}\left(27f_n + 102f_{n+\frac{1}{2}} + 72f_{n+1} + 42f_{n+\frac{3}{2}} - 3f_{n+2}\right),$$

$$y_{n+2} = y_n + \frac{h}{45}\left(7f_n + 32f_{n+\frac{1}{2}} + 12f_{n+1} + 32f_{n+\frac{3}{2}} + 7f_{n+2}\right).$$

2.4 Derivation of Continuous Three-Step method with Three Off-Step Points

In like manner, we consider here the derivation of continuous three-step method with three off-step points. Also, the three points here are selected in such a way of maintaining equally spaced interval.

By letting k = 8 and p = 3 in equation (2.8), we obtain

$$Y(x) = \sum_{r=0}^{7} a_r \varphi_r(x) \qquad (2.25)$$

Expanding (2.25), we have

$$Y(x) = a_0 + a_1\left(\frac{2(x-x_n}{3h} - 1\right) + a_2\left(\frac{5}{2}\left(\frac{2(x-x_n)}{3h} - 1\right)^2 - \frac{3}{2}\right) + a_3\left(\frac{7}{2}\left(\frac{2(x-x_n)}{3h} - 1\right)^3 - \frac{5}{2}\left(\frac{2(x-x_n)}{3h} - 1\right)\right) + a_4\left(\frac{63}{8}\left(\frac{2(x-x_n)}{3h} - 1\right)^4 - \frac{70}{8}\left(\frac{2(x-x_n}{3h} - 1\right)^2 + \frac{15}{8}\right) + a_5\left(\frac{99}{8}\left(\frac{2(x-x_n)}{3h} - 1\right)^5 - \frac{126}{8}\left(\frac{2(x-x_n}{3h} - 1\right)^3 + \frac{35}{8}\left(\frac{2(x-x_n}{3h} - 1\right)\right) + a_6\left(\frac{429}{16}\left(\frac{2(x-x_n)}{3h} - 1\right)^6 - \frac{693}{16}\left(\frac{2(x-x_n}{3h} - \right.\right.$$

$1\Big)^4 + \frac{315}{16}\left(\frac{2(x-x_n)}{3h} - 1\right)^2 - \frac{35}{16}\Big) + \quad a_7\left(\frac{715}{16}\left(\frac{2(x-x_n)}{3h} - 1\right)^7 - \frac{1287}{16}\left(\frac{2(x-x_n}{3h} - 1\right)^5 + \frac{693}{16}\left(\frac{2(x-x_n}{3h} - 1\right)^3 - \frac{105}{16}\left(\frac{2(x-x_n}{3h} - 1\right)\right)$. (2.26)

Expanding (2.26) and simplifying in terms of function t, we have

$$Y(t) = a_0 + a_1\left(\frac{2}{3}t - 1\right) + a_2\left(\frac{10}{9}t^2 - \frac{10}{3}t + 1\right) + a_3\left(\frac{28}{27}t^3 - \frac{14}{3}t^2 + \frac{16}{3}t - 1\right) + a_4\left(\frac{14}{9}t^4 - \frac{28}{3}t^3 + \frac{154}{9}t^2 - \frac{28}{3}t + 1\right) + a_5\left(\frac{44}{27}t^5 - \frac{110}{9}t^4 + 32t^3 - 55t^2 + \frac{265}{6}t - \frac{67}{4}\right) + a_6\left(\frac{572}{243}t^6 - \frac{572}{27}t^5 + \frac{638}{9}t^4 - \frac{968}{9}t^3 + 72t^2 - 18t + 1\right) + a_7\left(\frac{5720}{2187}t^7 - \frac{20020}{729}t^6 + \frac{9152}{81}t^5 - \frac{18590}{81}t^4 + \frac{6424}{27}t^3 - \frac{352}{3}t^2 + \frac{68}{3}t - 1\right), \quad (2.27)$$

where $t = \frac{x-x_n}{h}$.

Differentiating (2.27), we obtain

$$Y'(t) = \frac{2}{3}a_1 + a_2\left(\frac{20}{9}t - \frac{10}{3}\right) + a_3\left(\frac{28}{9}t^2 - \frac{28}{3}t + \frac{16}{3}\right) + a_4\left(\frac{56}{9}t^3 - 28t^2 + \frac{308}{9}t - \frac{28}{3}\right) + a_5\left(\frac{220}{27}t^4 - \frac{440}{9}t^3 + 96t^2 - 110t + \frac{265}{6}\right) + a_6\left(\frac{1144}{81}t^5 - \frac{2860}{27}t^4 + \frac{2552}{9}t^3 - \frac{968}{3}t^2 + 144t - 18\right) + a_7\left(\frac{40040}{2187}t^6 - \frac{40040}{243}t^5 + \frac{45760}{81}t^4 - \frac{74360}{81}t^3 + \frac{6424}{9}t^2 - \frac{704}{3}t + \frac{68}{3}\right). \quad (2.28)$$

Interpolating (2.27) at t = 0 and collocating (2.28) at t = $0\left(\frac{1}{2}\right)3$, we have a system of equations written in the matrix form AX = B as

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \frac{-67}{4} & 1 & -1 \\ 0 & \frac{2}{3} & \frac{-10}{3} & \frac{16}{3} & \frac{-28}{3} & \frac{265}{3} & -18 & \frac{68}{3} \\ 0 & \frac{2}{3} & \frac{-20}{9} & \frac{13}{9} & \frac{14}{9} & \frac{817}{108} & \frac{421}{162} & \frac{-9269}{17496} \\ 0 & \frac{2}{3} & \frac{-10}{9} & \frac{-8}{9} & \frac{28}{9} & \frac{-571}{54} & \frac{-398}{81} & \frac{4868}{2187} \\ 0 & \frac{2}{3} & 0 & \frac{-5}{3} & 0 & \frac{-343}{12} & 0 & \frac{-35}{8} \\ 0 & \frac{2}{3} & \frac{10}{9} & \frac{-8}{9} & \frac{-28}{9} & \frac{-2839}{54} & \frac{398}{81} & \frac{4868}{2187} \\ 0 & \frac{2}{3} & \frac{20}{9} & \frac{13}{9} & \frac{-14}{9} & \frac{-8255}{108} & \frac{-421}{162} & \frac{-9269}{17496} \\ 0 & \frac{2}{3} & \frac{10}{3} & \frac{16}{3} & \frac{28}{3} & \frac{-491}{6} & 18 & \frac{68}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ hf_{n+\frac{3}{2}} \\ hf_{n+2} \\ hf_{n+\frac{5}{2}} \\ hf_{n+3} \end{bmatrix}. \quad (2.29)$$

Solving equation (2.29) using Maple software, we obtain the value of the unknown parameters a_j, j = 0(1)7 as follows:

$$a_0 = y_n + \frac{h}{77000}\Big(71363f_n - 45738f_{n+\frac{1}{2}} - 198450f_{n+1} + 487220f_{n+\frac{3}{2}} - 191025f_{n+2} - 69498f_{n+\frac{5}{2}} + 61628f_{n+3}\Big),$$

$$a_1 = \frac{h}{15400}\Big(13391f_n - 14211f_{n+\frac{1}{2}} - 38925f_{n+1} + 102590f_{n+\frac{3}{2}} - 38925f_{n+2} - 14211f_{n+\frac{5}{2}} + 13391f_{n+3}\Big),$$

$$a_2 = \frac{h}{11000}\Big(3411f_n - 7386f_{n+\frac{1}{2}} - 12075f_{n+1} + 27540f_{n+\frac{3}{2}} - 12225f_{n+2} - 3306f_{n+\frac{5}{2}} + 4041f_{n+3}\Big),$$

$$a_3 = \frac{h}{80080}\Big(2219f_n - 11880f_{n+\frac{1}{2}} - 16011f_{n+1} + 3824f_{n+\frac{3}{2}} - 16011f_{n+2} + 11880f_{n+\frac{5}{2}} + 2219f_{n+3}\Big),$$

(2.30)

$$a_4 = \frac{h}{7280}\Big(-183f_n - 48f_{n+\frac{1}{2}} + 645f_{n+1} - 645f_{n+2} + 48f_{n+\frac{5}{2}} + 183f_{n+3}\Big),$$

$$a_5 = \frac{h}{3850}\Big(69f_n - 99f_{n+\frac{1}{2}} - 225f_{n+1} + 510f_{n+\frac{3}{2}} - 225f_{n+2} - 99f_{n+\frac{5}{2}} + 69f_{n+3}\Big),$$

$$a_6 = \frac{h}{2800}\Big(-27f_n + 108f_{n+\frac{1}{2}} - 135f_{n+1} + 135f_{n+2} - 108f_{n+\frac{5}{2}} + 27f_{n+3}\Big),$$

$$a_7 = \frac{h}{50050}\Big(243f_n - 1458f_{n+\frac{1}{2}} + 3645f_{n+1} + 4860f_{n+\frac{3}{2}} + 3645f_{n+2} - 1458f_{n+\frac{5}{2}} + 243f_{n+3}\Big).$$

Substituting (2.30) into (2.28) yields a continuous implicit hybrid three-step method in the form

$$y(t) = \alpha_j(t)y_n + h\left(\sum_{j=0}^{3}\beta_j(t)f_{n+j} + \beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(t)f_{n+\frac{3}{2}} + \beta_{\frac{5}{2}}(t)f_{n+\frac{5}{2}}\right), \quad (2.31)$$

where $\alpha_j(t)$ and $\beta_j(t)$ are continuous coefficients obtained as

$$\alpha_0(t) = 1,$$

$$\beta_0(t) = \frac{4}{315}t^7 - \frac{7}{45}t^6 + \frac{7}{9}t^5 - \frac{49}{21}t^4 + \frac{406}{135}t^3 - \frac{49}{20}t^2 + t,$$

$$\beta_{\frac{1}{2}}(t) = \frac{-8}{105}t^7 + \frac{8}{9}t^6 - \frac{62}{15}t^5 + \frac{29}{3}t^4 - \frac{58}{5}t^3 + 6t^2,$$
$$\beta_1(t) = \frac{4}{21}t^7 - \frac{19}{9}t^6 + \frac{137}{15}t^5 - \frac{461}{24}t^4 + \frac{39}{2}t^3 - \frac{15}{2}t^2,$$
$$\beta_{\frac{3}{2}}(t) = \frac{-16}{63}t^7 + \frac{8}{3}t^6 - \frac{484}{45}t^5 + \frac{62}{3}t^4 - \frac{508}{27}t^3 + \frac{20}{3}t^2,$$
$$\beta_2(t) = \frac{4}{21}t^7 - \frac{17}{6}t^6 + \frac{107}{15}t^5 - \frac{307}{24}t^4 + 11t^3 - \frac{15}{4}t^2,$$
$$\beta_{\frac{5}{2}}(t) = \frac{-8}{105}t^7 + \frac{32}{45}t^6 - \frac{38}{15}t^5 + \frac{13}{3}t^4 - \frac{18}{5}t^3 + \frac{6}{5}t^2,$$
$$\beta_3(t) = \frac{4}{315}t^7 - \frac{1}{9}t^6 + \frac{17}{45}t^5 - \frac{5}{8}t^4 + \frac{137}{270}t^3 - \frac{1}{6}t^2.$$

Evaluating (2.31) at $x_{n+\frac{1}{2}}$, x_{n+1}, $x_{n+\frac{3}{2}}$, x_{n+2}, $x_{n+\frac{5}{2}}$ and x_{n+3}, the following discrete schemes are obtained:

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{120960}\Big(19087f_n + 65112f_{n+\frac{1}{2}} - 46461f_{n+1} + 37504f_{n+\frac{3}{2}} - 20211f_{n+2} + 6312f_{n+\frac{5}{2}} - 863f_{n+3}\Big),$$

$$y_{n+1} = y_n + \frac{h}{7560}\Big(1139f_n + 5640f_{n+\frac{1}{2}} + 33f_{n+1} + 1328f_{n+\frac{3}{2}} - 807f_{n+2} + 264f_{n+\frac{5}{2}} - 37f_{n+3}\Big),$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{4480}\Big(685f_n + 3240f_{n+\frac{1}{2}} + 1161f_{n+1} + 2176f_{n+\frac{3}{2}} - 729f_{n+2} + 216f_{n+\frac{5}{2}} - 29f_{n+3}\Big), \quad (2.32)$$

$$y_{n+2} = y_n + \frac{h}{945}\Big(143f_n + 696f_{n+\frac{1}{2}} + 192f_{n+1} + 752f_{n+\frac{3}{2}} + 87f_{n+2} + 24f_{n+\frac{5}{2}} - 4f_{n+3}\Big),$$

$$y_{n+\frac{5}{2}} = y_n + \frac{h}{24192}\Big(3715f_n + 17400f_{n+\frac{1}{2}} + 6375f_{n+1} + 16000f_{n+\frac{3}{2}} + 11625f_{n+2} + 5640f_{n+\frac{5}{2}} - 275f_{n+3}\Big),$$

$$y_{n+3} = y_n + \frac{h}{280}\Big(41f_n + 216f_{n+\frac{1}{2}} + 27f_{n+1} + 272f_{n+\frac{3}{2}} + 27f_{n+2} + 216f_{n+\frac{5}{2}} + 41f_{n+3}\Big).$$

3.0 Analysis of the Methods

Basic properties of the methods are analyzed to establish their validity. These properties, namely; order, error constant, consistency and zero stability reveal the nature of convergence of the methods. In what follows, a brief introduction of these properties is made for a better comprehension of the Section.

3.1 Order and Error Constant

The linear difference operator L associated with the continuous implicit multistep method developed is defined as:

$$\sum_{j=0}^{k} \alpha_j(x) y(x_{n+j}) = h \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}) \qquad (3.1)$$

with

$$\mathcal{L}[y(x);h] = \sum_{j=0}^{k} \left[\alpha_j(x) y(x+jh) - h\beta_j(x) y'(x+jh)\right] = 0, \qquad (3.2)$$

where y(x) is an arbitrary test function that is continuously differentiable in the interval [a,b].

The Taylor's series expansion about the point x gives

$$\mathcal{L}[y(x);h] = C_0 h y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^p(x). \qquad (3.3)$$

The difference operator L and the associated continuous implicit hybrid methods are of order p if $C_0 = C_1 = C_2 = \cdots C_p = 0$ *and* $C_{p+1} \neq 0$. The term $C_{p+1} \neq 0$ is called the error constant. The order and the error constants of the main methods and block methods are presented below:

Table 3.1: The order and the error constants of the main methods and block methods

Step number	Main Method	Order	Error Constant
1	(2.15)	4	$\frac{-1}{2880}$
2	(2.24d)	6	$\frac{-1}{15120}$
3	(2.32f)	8	$\frac{-9}{716800}$

Table 3.2: The order and the error constants of the main methods and block methods

Step number	Main Method	Order	Error Constant
1	(2.16)	3	$\frac{1}{384}$
2	(2.24a)	5	$\frac{3}{10240}$
	(2.24b)	5	$\frac{1}{5760}$
	(2.24c)	5	$\frac{3}{10240}$
3	(2.32a)	7	$\frac{275}{6193152}$
	(2.32b)	7	$\frac{1}{30240}$
	(2.32c)	7	$\frac{9}{229376}$
	(2.32d)	7	$\frac{1}{30240}$
	(2.32e)	7	$\frac{275}{6193152}$

3.2 Consistency of the Methods

The concept consistency is related to the convergence of the multistep methods in the sense that it controls the magnitude of the local truncation error committed at every integration step. According to Fatunla (1991), the linear multistep method (3.1) is said to be **consistent** if it has order $p \geq 1$. The order of all the schemes derived have been investigated in the preceding section to satisfy this condition i.e. they all have order $p > 1$. This shows that all the schemes derived are consistent.

3.3 Zero Stability of the Methods

The stability property of a method reveals the extent at which a method copes with a problem for a given step-length h. To analyze the methods for zero-stability, we use vector notation and the matrices

$$\boldsymbol{A} = (a_{ij}), \quad \boldsymbol{B} = (b_{ij}), \text{ column vectors } \boldsymbol{e} = (e_1, \ldots . e_r)^T, \ \boldsymbol{d} = (d_1, \ldots . d_r)^T,$$

$$\boldsymbol{y_m} = (y_{n+1}, \ldots . y_{n+r})^T, \qquad \boldsymbol{F}(\boldsymbol{y_m}) = (f_{n+1}, \ldots . f_{n+r})^T$$

and write them as block method given by

$$Ay_m = hBF(y_m) + ey_n + hdf_n \tag{3.4}$$

where h is a fixed mesh size within a block.

In equations (2.15) and (2.16)

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad e=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}1/6 & 2/3\\ -1/24 & 1/3\end{bmatrix}\quad \text{and}\quad d=\begin{bmatrix}0 & 1/6\\ 0 & 5/24\end{bmatrix}.$$

The first characteristic polynomial of the block method is given by

$$\rho(R)=\det(RA^0-A^1), \qquad (3.5)$$

where

$$A^0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \text{and}\quad A^1=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}.$$

Substituting A^0 *and* A^1 in equation (3.5) and solving for R, the values of R are obtained as 0 and 1. According to [8], the block method equations (2.15) and (2.16) are zero-stable, since from (3.5), $\rho(R)=0$, satisfy $|R_j|\leq 1$, j = 1,...,k, and for those roots with $|R_j|=1$, the multiplicity does not exceed one.

Considering equation (2.24) and arranging it in accordance with equation (3.4), we have

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad e=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}4/15 & 32/45 & 32/45 & 7/45\\ -11/60 & 53/720 & 323/720 & -19/1440\\ 2/15 & 1/45 & 31/45 & -1/180\\ 9/20 & 21/80 & 51/80 & -3/160\end{bmatrix}$$

and

$$d=\begin{bmatrix}0 & 0 & 0 & 7/45\\ 0 & 0 & 0 & 251/1440\\ 0 & 0 & 0 & 29/180\\ 0 & 0 & 0 & 27/160\end{bmatrix}.$$

The first characteristic polynomial of the block method is given by

$$\rho(R)=\det\,(RA^0-A^1), \qquad (3.6)$$

where

$$A^0=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \text{and}\quad A^1=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Substituting A^0 *and* A^1 in equation (3.6) and solving for R, the values of R are obtained as 0 and 1. According to Fatunla (1991), the block method equations (2.24) are zero-stable, since from (3.6), $\rho(R) = 0$, satisfy $|R_j| \le 1$, j = 1,...,k, and for those roots with $|R_j| = 1$, the multiplicity does not exceed one.

In like manner, the following are obtained from equation (2.32).

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 41/280 & 27/35 & 27/280 & 34/35 & 27/280 & 27/35 \\ -863/120960 & 263/5040 & -6737/40320 & 293/945 & -15487/40320 & 2713/5040 \\ -37/7560 & 11/315 & -269/2520 & 166/945 & 11/2520 & 47/63 \\ -29/4480 & 27/560 & -729/4480 & 17/35 & 1161/4480 & 81/122 \\ -4/945 & 8/315 & 29/315 & 752/945 & 64/315 & 232/315 \\ -275/24192 & 235/1008 & 3875/8064 & 125/189 & 2125/8064 & 725/1008 \end{bmatrix}$$

and $$d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 41/280 \\ 0 & 0 & 0 & 0 & 0 & 19087/120960 \\ 0 & 0 & 0 & 0 & 0 & 1139/7560 \\ 0 & 0 & 0 & 0 & 0 & 137/896 \\ 0 & 0 & 0 & 0 & 0 & 143/945 \\ 0 & 0 & 0 & 0 & 0 & 3715/24192 \end{bmatrix},$$

where in (3.6), A^0 *and* A^1 are

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Substituting A^0 *and* A^1 in equation (3.6) and solving for R, the values of R are obtained as 0 and 1. According to Fatunla (1991), the block method equations (2.32) are zero-stable, since from (3.6), $\rho(R) = 0$, satisfy $|R_j| \le 1$, j = 1,...,k, and for those roots with $|R_j| = 1$, the multiplicity does not exceed one.

3.4 Convergence

According to the fundamental theorem of Dahlquist (1979), the necessary and sufficient conditions for convergence are consistence and zero stability. All the schemes derived satisfy the conditions for consistency and stability. Hence, the schemes are convergent.

3.5 Numerical Examples

Here, we consider the application of the derived schemes to three test problems for the efficiency and accuracy of the methods implemented as block methods.

Problem 3.1: (A constant coefficient nonlinear homogeneous problem)

$$y' + y^2 = 0,\ y(0) = 1,\ h = 0.1,$$

$$Exact\ Solution: \quad y(x) = \frac{1}{x+1}.$$

Problem 3.2: (A variable coefficient linear homogeneous problem)

$$y' - xy = 0, \quad y(0) = 1, \quad h = 0.01,$$

$$Analytical\ Solution: \quad y(x) = e^{0.5x^2}.$$

Problem 3.3: (A constant coefficient nonhomogeneous linear problem)

$$y' + y = x, \quad y(0) = 0, \quad h = 0.1,$$

$$True\ Solution: \quad y(x) = x + e^{-x} - 1.$$

3.6 Tables of Results

Table 3.3a: Numerical Results of IHS for **Problem 3.1**

X	EXACT	ISIHM	2SIHM	3SIHM
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9090909091	0.9090911939	0.9090908517	0.9090909054
0.2	0.8333333333	0.8333337388	0.8333333466	0.8333333303
0.3	0.7692307692	0.7692312161	0.7692307624	0.7692307702
0.4	0.7142857143	0.7142861638	0.7142857276	0.7142857147
0.5	0.6666666667	0.6666671002	0.6666666715	0.6666666670
0.6	0.6250000000	0.6250004093	0.6250000113	0.6250000007

0.7	0.5882352941	0.5882356762	0.5882353013	0.5882352946
0.8	0.5555555556	0.5555559101	0.5555555649	0.5555555560
0.9	0.5263157895	0.5263161175	0.5263157966	0.5263157900
1.0	0.5000000000	0.500000303	0.5000000077	0.5000000005

The exact solutions and the computed results from the proposed methods for **problem 3.1**

Table 3.3b: Error of IHS for **Problem 3.1**

X	Error in 1SIHM	Error in 2SIHM	Error in 3SIHM	Error in Areo (2011)
0.1	2.848E-07	5.740E-08	3.700E-09	2.400E-04
0.2	4.055E-07	1.330E-08	3.000E-09	5.600E-04
0.3	4.469E-07	6.800E-09	1.000E-09	7.100E-04
0.4	4.495E-07	1.330E-08	4.000E-10	8.400E-04
0.5	4.335E-07	4.800E-09	3.000E-10	9.600E-04
0.6	4.093E-07	1.130E-08	7.000E-10	1.100E-04
0.7	3.821E-07	7.200E-09	5.000E-10	1.100E-03
0.8	3.545E-07	9.300E-09	4.000E-10	1.300E-03
0.9	3.280E-07	7.100E-09	5.000E-10	1.500E-03
1.0	3.030E-07	7.700E-09	5.000E-10	1.600E-02

Comparing the absolute errors in the new methods to errors in Areo (2011) for Problem 3.1

Table 3.4a: Numerical Results of IHS for **Problem 3.2**

X	EXACT	ISIHM	2SIHM	3SIHM

0.00	1.000000000	1.000000000	1.000000000	1.000000000
0.01	1.000050001	1.000050001	1.000050001	1.000050001
0.02	1.000200020	1.000200020	1.000200020	1.000200020
0.03	1.000450101	1.000450101	1.000450101	1.000450101
0.04	1.000800320	1.000800320	1.000800320	1.000800320
0.05	1.001250782	1.001250781	1.001250781	1.001250782
0.06	1.001801621	1.001801620	1.001801621	1.001801621
0.07	1.002453004	1.002453003	1.002453003	1.002453004
0.08	1.003205125	1.003205125	1.003205124	1.003205125
0.09	1.004058212	1.004058212	1.004058212	1.004058211
0.10	1.005012521	1.005012521	1.005012520	1.005012520
0.11	1.006068338	1.006068338	1.006068338	1.006068338
0.12	1.007225982	1.007225982	1.007225982	1.007225982
0.13	1.008485802	1.008485802	1.008485802	1.008485802
0.14	1.009848177	1.009848177	1.009848177	1.009848177
0.15	1.011313519	1.011313519	1.011313519	1.011313519
0.16	1.012882271	1.012882270	1.012882270	1.012882271
0.17	1.014554906	1.014554905	1.014554906	1.014554906
0.18	1.016331931	1.016331931	1.016331931	1.016331931
0.19	1.018213886	1.018213885	1.018213885	1.018213885
0.20	1.020201340	1.020201339	1.020201340	1.020201339

The exact solutions and the computed results from the proposed methods for **problem 3.2**

Table 3.4b: Error of IHS for **Problem 3.2**

X	Error in ISIHM	Error in 2SIHM	Error in 3SIHM	Error in Adeniyi *et al*

				(2008)
0.01	0.000000E+00	0.000000E+00	0.000000E+00	9.044090E-4
0.02	0.000000E+00	0.000000E+00	0.000000E+00	1.607698E-3
0.03	0.000000E+00	0.000000E+00	0.000000E+00	2.109971E-3
0.04	0.000000E+00	0.000000E+00	0.000000E+00	2-411305E-3
0.05	1.000000E-09	1.000000E-09	0.000000E+00	2.511744E-3
0.06	1.000000E-09	0.000000E+00	0.000000E+00	2-411304E-3
0.07	9.999999E-10	9.999999E-10	0.000000E+00	2.109970E-3
0.08	0.000000E+00	1.000000E-09	0.000000E+00	1.607697E-3
0.09	0.000000E+00	0.000000E+00	9.999999E-10	9.044090E-4
0.10	0.000000E+00	1.000000E-09	1.000000E-09	0.000000
0.11	0.000000E+00	0.000000E+00	0.000000E+00	9.317440E-4
0.12	0.000000E+00	0.000000E+00	0.000000E+00	1.656287E-3
0.13	0.000000E+00	0.000000E+00	0.000000E+00	2.173737E-3
0.14	0.000000E+00	0.000000E+00	0.000000E+00	2.484176E-3
0.15	0.000000E+00	0.000000E+00	0.000000E+00	2.587649E-3
0.16	1.000000E-09	1.000000E-09	0.000000E+00	2.484171E-3
0.17	1.000000E-09	0.000000E+00	0.000000E+00	2.173734E-3
0.18	0.000000E+00	0.000000E+00	0.000000E+00	1.656284E-3
0.19	1.000000E-09	1.000000E-09	1.000000E-09	9.317430E-4
0.20	1.000000E-09	0.000000E+00	1.000000E-09	0.000000

Comparing the absolute errors in the new methods to errors in Adeniyi (2008) for **Problem 3.2**

Table 3.5a: Numerical Results of IHS for **Problem 3.3**

X	EXACT	ISIHM	2SIHM	3SIHM
0.0	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.004837418036	0.004837430611	0.004837417896	0.004837418037
0.2	0.018730753080	0.018730775840	0.018730753110	0.018730753080
0.3	0.040818220680	0.040818251580	0.040818220600	0.040818220680
0.4	0.070320046040	0.070320083310	0.070320046090	0.070320046040
0.5	0.106530659700	0.106530701900	0.106530659700	0.106530659700
0.6	0.148811636100	0.148811681900	0.148811636200	0.148811636100
0.7	0.196585303800	0.196585352100	0.196585303800	0.196585303800
0.8	0.249328964100	0.249329014100	0.249328964200	0.249328964100
0.9	0.306569665970	0.306569710600	0.306569659800	0.306569659700
1.0	0.367879441200	0.367879492300	0.367879441300	0.367879441100

Showing the exact solutions and the computed results from the proposed methods for **problem 3.3**

Table 3.5b: Error of IHS for **Problem 3.3**

X	Error in ISIHM	Error in 2SIHM	Error in 3SIHM	Error in Areo (2011)
0.1	1.257E-08	1.400E-10	1.000E-12	0.000
0.2	2.276E-08	3.000E-11	0.000E+00	0.000
0.3	3.090E-08	8.000E-11	0.000E+00	6.000E-10
0.4	3.727E-08	5.000E-11	0.000E+00	2.000E-11
0.5	4.220E-08	0.000E+00	0.000E+00	7.000E-10
0.6	4.580E-08	1.000E-10	0.000E+00	1.000E-10
0.7	4.830E-08	0.000E+00	0.000E+00	8.000E-10
0.8	5.000E-08	1.000E-10	0.000E+00	2.000E-10
0.9	4.463E-08	6.170E-09	6.270E-09	9.000E-10
1.0	5.110E-08	1.000E-10	1.000E-10	4.000E-10

Comparing the absolute errors in the new methods to errors in Areo (2011) for **Problem 3.3**

4. Conclusion

In this work, the derivation of continuous hybrid schemes for the numerical solution of first order IVPs in ODEs have developed through collocation and interpolation technique with orthogonal polynomials of weight function $w(x) = x^2$ as basis function. The derived schemes are implemented on three test problems to test the applicability, efficiency and accuracy of the schemes. The schemes, when compared with existing methods, compete favourably well and behave like theoretical solution.

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