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Numerical Approximation of Fractional Integro-differential Equations by an Iterative Decomposition Method

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Abstract

In this paper, an attempt is made to approximate the solution of linear and nonlinear fractional integro-differential equations, by applying an iterative decomposition method. The approximate solution of each problem is presented as a rapidly convergent series of easily computable terms. Results obtained are compared favourably with known results to illustrate the accuracy and efficiency of the method.

Keywords: Fractional Integro Differential Equation, Iterative Decomposition Method, Approximation, Accuracy, Error

1. Introduction

In recent years, there has been continuously renewed interest in integro-differential equations. Many mathematical models of physical phenomena produce integro-differential equations e.g fluid dynamics, biological models and chemical kinetics [3]. Electromagnetism, acoustics, viscoelasticity, electrochemistry and material science are also well-described by fractional integro-differential equations [2, 5].

Owing to the numerous applications of fractional calculus, in diverse fields, the solution techniques for fractional differential equations of various forms and classes, continue to attract

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growing interest from many researchers. Efficient numerical methods are being developed by many authors, some of such algorithms have been based on known numerical techniques like quadrature and extrapolation, as well as the use of Mittag-Leffler functions and other special functions for approximation [3, 4, 11].

In the present study, fractional derivatives are understood in the Caputo sense, although, there are several approaches to the generalization of fractional derivatives, the Caputo derivative is the most suitable for real-life physical problems. Some other approaches are Riemann-Liouville, Grunwald-Letnikov, and Weyl derivatives [2, 3, 6, 9].

Some well-known approximation techniques which have been successful with integer-order integro-differential equations have been modified and applied to fractional integro-differential equations. The Adomian Decomposition Method (ADM) was applied to solve fractional integro-differential equations in [9]. The Differential Transform Method (DTM) was modified for fractional indices in [1] and applied for fractional integro-differential equations. The Homotopy Perturbation Method (HPM) was applied in [4, 5], while in [10] the solutions by Variational Iteration Method (VIM) and the HPM were compared.

In this paper, we apply an Iterative Decomposition Method (IDM), which had been applied to integer- order differential equations [13].The method is modified to appropriate both linear and nonlinear fractional integro-differential equations.

The layout of the paper is as follows: In section 2, we give some very vital definitions which are essential for the understanding of the problem. In section 3, we present the Iterative Decomposition Method, while in section 4, the method is applied to solve some examples, and conclusions are drawn in section 5.

2. Definitions

We shall review some basic definitions of fractional calculus, which are essential for the proper understanding of the problems of fractional differential equations.

Definition 2.1. (Caputo Derivative): The Caputo fractional derivative of $f(x)$ of order $\alpha>0$ is defined as

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt,\ n-1<\alpha\le n, n\in\mathbb{R}\,. \quad (1)$$

Definition 2.2: A function $f(x), x>0$ is said to be in the space $C_{\mu,}\mu\epsilon\mathbb{R}$ if there exists $p\in\mathbb{R}, p>\mu$ such that$f(x0 = \ x^p f_1(x)$, where $f_1(x)\in C[0,x]$. Clearly, $C_\mu \subset C_\beta\ if\ \beta\le\mu$.

Definition 2.3: A function $f(x), x>0$ is said to be in the space $C_\mu^m, m\in\mathbb{N}\cup\{0\} if\ f^{(m)}(x)\in C_\mu$.

Definition 2.4 The Riemann-Liouville fractional integral operator of order $\alpha\ge 0$, of a function $f\in C_{\mu,}\ ,\mu\ge -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\ \alpha>0,\ \alpha>0, x>0 \quad (2)$$

Properties of the operator J^α can be found in [11] and include the following

$$J^\alpha f(x) = f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta}f(x)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

Also, if $m-1<\alpha<m, m\in\mathbb{N}\ and\ f\in C_\mu^m, \mu\ge -1, then$

$$D^\alpha J^\alpha f(x) = f(x) \quad (3)$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{n=0}^{m-1}\frac{x^n}{n!}f^{(n)}(0)\,. \quad (4)$$

The general fractional integro-differential equation is of the form

$$D_*^\alpha y(x) = a(x)y(x) + f(x) + \int_0^x K(t,s)F\big(y(s)\big)ds \quad (5)$$

$$y(0) = \beta, \alpha<1,$$

where D_*^α is the Caputo fractional derivative and α is a parameter.

3. Iterative Decomposition Method

From [13] the Iterative Decomposition Method suggests that by applying the inverse operator $D^{-\alpha} = J^{\alpha}$ which is the inverse of D_*^{α} to both sides of (5), we have

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} y^{(k)}(0) + D^{-\alpha}\{a(x)y(x) + f(x)\} + D^{-\alpha}\{\int_0^x K(t,s)F(y,s)ds\}. \qquad (6)$$

The IDM suggests further that the solution is decomposed into the infinite series of convergent terms

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \qquad (7)$$

Taking

$$y_0 = \sum_{k=0}^{m-1} \frac{x^k}{k!} y^{(k)}(0) + D^{-\alpha}\{a(x)y(x) + f(x)\}\ . \qquad (8)$$

Then, we have

$$y_{n+1} = D^{-\alpha}\{[\sum_{j=o}^{n} \int_0^x K(t,s)F\left(y_j(s)\right)ds] - [\sum_{j=0}^{n-1} K(t,s)F\left(y_j(s)\right)ds]\}. \qquad (9)$$

From (9), we can approximate the solution by

$$\Phi_N(x) = \sum_{i=0}^{N-1} y_i \qquad (10)$$

and $\lim_{N\to\infty} \Phi_N(x) = \sum_{i=0}^{N-1} y_i$. (11)

4. Numerical Experiment

We now apply the method proposed in section 3 to some numerical examples, to establish the accuracy and efficiency of the method.

Example 4.1: Consider the nonlinear fractional integro-differential equation [5, 9]

$$D^{\alpha}y(x) = 1 + \int_0^x y(t)D^{\alpha}y(t)dt,\ 0 \le x < 1,\ \ 0 \le \alpha \le 1 \qquad (12)$$

with initial condition $y(0) = 0$.

For $\alpha = 1$, the exact solution is

$$y(x) = \sqrt{2}\tan\left(\frac{\sqrt{2}}{3}x\right). \tag{13}$$

By using the inverse operator $D^{-\alpha}$ on both sides of (12), we have

$$y(x) = D^{-\alpha}(1) + D^{-\alpha}\{\int_0^x y(t)D^{\alpha}y(t)dt\}. \tag{14}$$

Taking $y_0 = D^{-\alpha}(1)$, we have

$$y_0(x) = \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \tag{15}$$

$$y_1(x) = \frac{1}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+2)}x^{2\alpha+1}$$

$$= \frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+2)}x^{2\alpha+1} \tag{16}$$

$$y_2(x) = \left[\frac{1}{(\Gamma(\alpha+1)^3} + \frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+2)}\right]\left[\frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+2)}\right]x^{3\alpha+2}. \tag{17}$$

Then, y(x) can be approximated as

$$y(x) = \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+2)}x^{2\alpha+1} + \left\{\frac{\Gamma(2\alpha+2)+[\Gamma(\alpha+1)][\Gamma(\alpha+2)]}{(\alpha+1)[\Gamma(\alpha+1)]^2\Gamma(2\alpha+2)}\right]\left[\frac{2\alpha+2}{\Gamma(3\alpha+3)}\right]x^{3\alpha+2}. \tag{18}$$

Setting $\alpha = 1$, we have

$$y(x) = x + \frac{\Gamma(3)}{2\Gamma(2)\Gamma(4)}x^3 + \left(\frac{\Gamma(4)+\Gamma(2)\Gamma(3)}{2(\Gamma(2))^2\Gamma(4)}\right)\left(\frac{4}{\Gamma(6)}\right)x^5. \tag{19}$$

Table 1: Error of example 4.1

X	Exact Solution	IDM Approx.	Error
0.0	0.0	0.0	0.0
0.1	0.1001670007	0.1001668889	1.118E-7
0.2	0.2013440871	0.201340444	3.643E-6
0.3	0.3045825028	0.30455400	2.850E-5
0.4	0.411017423	0.410594222	4.232E-4
0.5	0.5219305155	0.5215277778	4.027E-4

0.6	0.6387957043	0.637728000	1.068E-3
0.7	0.7633858023	0.7609015556	2.484E-3
0.8	0.8978815375	0.8926151111	5.266E-3
0.9	1.045043135	1.0342622	1.078E-2
1.0	1.208460242	1.17777778	3.068E-2

Table 1compares the exact solution for $\alpha = 1$ with the approximate solution of Example 4.1 obtained by IDM.

Note that $Error = |Approx. soln. - Exact\ soln.|$

Table 2

x	y(x) for $\alpha =$ 0.5	y(x) for $\alpha =$ 0.75	y(x) for $\alpha =$ 0.9
0.0	0.0	0.0	0.0
0.1	0.362262217	o.194437395	0.131192521
0.2	0.529467729	0.330880936	0.246358346
0.3	0.683005395	0.456455955	0.358447857
0.4	0.848270069	0.579673363	0.478522735
0.5	1.04213390	0.705240779	0.585841026
0.6	1.27974996	0.836729573	0.705380584
0.7	1.57619185	0.977344507	0.831400123
0.8	1.94699791	1.30237881	0.965916811
0.9	4.40837766	1.29866625	1.11111081
1.0	2.97729140	1.48608018	1.26937670

Solutions of Example 4.1 for α=0.5, 0.75 and 0.9.

Example 4.2: Consider the nonlinear fractional integro-differential equation [9, 12]

$$D_*^{0.9} y(x) = -1 + \int_0^x y^2(t)dt, 0 \leq x \leq 1 \tag{20}$$

with initial condition $y(0) = 0$. Applying the inverse operator $D^{-0.9}$ to both sides of (20), we have

$$y(x) = D^{-0.9}(-1) + D^{-0.9}\{\int_0^x y^2(t)dt\} \quad (21)$$

$$= \frac{1}{\Gamma(0.9)} \int_0^x (x-t)^{0.1} dt + D^{-0.9}\{\int_0^x y^2(t)dt\}$$

$$= -1.008694635x^{0.9} + D^{-0.9}\{\int_0^x y^2(t)dt\}. \quad (22)$$

Then, y(x) can be approximated as

$$y(x) = -1.039717197x^{0.9} + 0.1017155725x^{3.7} - 0.004345205712x^{5.5}. \quad (23)$$

In table 3 below, we compare our result with the result for the same problem in [5].

Table 2: Results obtained by HPM [5] and IDM

X	HPM [5]	IDM
0.0	0.0	0.0
0.125	-0.15997	-0.15996
0.25	-0.29790	-0.29798
0.375	-0.42689	-0.42690
0.5	-0.54824	-0.54844
0.625	-0.66086	-0.66193
0.75	-0.76327	-0.76788
0.875	-0.85359	-0.85809
1.0	-0.92988	-0.92935

Example 4.3 : Consider the initial value problem consisting of the multi-fractional order integro-differential equation [7]

$$D_*^{0.5} y(x) = \frac{6}{\Gamma(5.5)} x^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)} x^{4.5} + J^{1.5} y(x), \ x \epsilon [0, 1] \quad (24)$$

with initial condition $y(0) = 0$. The exact solution of the problem is $y(x) = x^3$.

By applying the IDM to (24) in Example 4.3, we found that y(x) is approximated as

$$y(x) = 1.00000445x^3 + (8.8623E - 5)x^5 - (2.3809E - 7)x^7.$$

Approximate solution obtained after two iterations.

Table 3: Errors obtained for Example 4.3

X	Exact Solution	IDM Approx.	Error
0.0	0.0	0.000000000000	0,0
0.1	1.00E-3	1.000003584E-3	3.5637E-9
0.2	8.00E-3	8.000063956E-3	7.2376E-9
0.3	2.70E-3	2.700033545E-3	3.3545E-7
0.4	6.40E-2	0.063999376910	6.2309E-7
0.5	1.25E-1	0.125003323900	3.3239E-6
0.6	2.16E-1	0.216007845900	7.8459E-6
0.7	3.43E-1	0.343016401600	1.6402E-5
0.8	5.12E-1	0.512031268500	3.1268E-5
0.9	7.29E-1	0.729055461200	5.5461E-5
1.0	1.000000	1.000092835000	9.2835E-5

3. Conclusion

From the examples given, the Iterative Decomposition Method (IDM) proved to be very efficient in the solution of fractional integro-differential equations. The solution of Example 4.1 by IDM is very close to the exact solution, even for very few terms of the approximating series. Example 4.2 shows that the method gives solutions that are comparable to known and tested methods, in terms of accuracy and efficiency. Furthermore, for cases where the exact solutions are unknown, the method is a useful tool for approximating solutions. The strength of the IDM includes the fact that we do not require to find some polynomials, unlike in the case of the ADM. Neither do we require rigorous or elaborate mathematical details.

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