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Solution of System of Higher Order Integro-Differential Equations by Perturbed Variational Iteration Method

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Abstract

Variational Iteration Method (VIM) is a numerical method for solving a wide class of non-linear problems, first envisioned by the Chinese Mathematician He (1998). In this paper, higher order integro differential equations are reduced to a system of integral equations. The reduced system is then perturbed by using Chebyshev Polynomial and solved by Variational Iteration method. The results obtained for some illustrative examples showed that the perturbed variational iteration method is efficient and reliable. Examples are given to illustrate the efficiency and implementation of the method.

Keywords: Integro-differential equations, Variational Iteration Method, Integral equations

1. Introduction

Nonlinear phenomena, that appear in many applications in scientific fields, such as Fluid Dynamics, Solid State Physics, Mathematical biology and Chemical Kinetics, can be modeled by integral equations. Integro-differential equations (IDE) play a prominent role in many branches of linear and nonlinear functional analysis and their application. Higher order Integro-differential equations arise in Mathematical, applied and Engineering Sciences, Astrophysics, Solid state physics, Astronomy, fluid Dynamics, Beam theory and Chemical reaction diffusion models (Najafzadeh et al., 2012). Variational Iteration Method (He, 1997, 1999, 2007) is a powerful device for solving various kinds of equations, linear and nonlinear. The method has successfully been applied to many situations. For example, He (2007) used the method to solve some integro-differential equations where he chose initial approximate solution in the form of exact solution with unknown constants.

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Abbasbandy and Shivanion (2009) used VIM to solve systems of nonlinear Volterra's Integro-differential equations.

Najafzadeh et al. (2012) transformed higher order IDEs into a system of integral equations and then solved by VIM. Salehpoor et al. (2010) presented a modification of VIM and applied it to systems of linear and nonlinear ODEs. Biazar et al. (2010) employed VIM to solve linear and nonlinear system of IDEs. In this paper, we considered the reduction of higher order integro-differential equation to a system of first order integro-differential equations, since every ODE of order n can be written as a system consisting of n ordinary differential equations of order one. The basic motivation here is to get a better approximation.

2. Variational Iteration Method:

Variational Iteration Method (VIM) is based on the general Lagranges's multiplier method (Inokuti et al., 1978). The main feature of the method is that the solution of a Mathematical problem with linearization assumption is used as initial approximation. Then a more highly precise approximation at some special point can be obtained. To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation

$$Lu + Nu = g(x), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to VIM (He 1999, 2000, 2006), we can construct a correction functional as follows

$$u_{n+1}(x) = \mathrm{u}_n(x) + \int_\circ^x \lambda\{Lu_n(T) + N\bar{u}_n(T) - g(T)\}dT,\ n \geq 0, \tag{2.2}$$

λ is a general langrangian multiplier (Inokuti et al, 1978) which can be identified optimally via Variational theory. Subscript n denotes the nth-order approximation, $\bar{u}_n$ is considered as a restricted variation (He, 1999, 2000) i.e. $\delta\bar{u}_n = 0$.

3. Solution Techniques

To convey the idea of the technique, we considered the following system of differential equations (Najafzadeh, 2012)

$$x'_i(t) = f_i(t, x_i), \bar{u}_n i = 1, 2, 3, \cdots,\ n, \tag{3.1}$$

subject to the boundary conditions

$$x_i(0) = c_i, \bar{u}_n i = 1,2,3,\cdots,n \,. \tag{3.2}$$

Rewriting the system (3.1) in the form

$$x'_i(t) = f_i(t) + g_i(t), i = 1,2,3,\cdots,n \tag{3.3}$$

subject to the boundary conditions

$$x_i(0) = c_i, i = 1,2,3,\cdots,n$$

and g_i is as defined in (2.1). The correction functional for the nonlinear system

(3.1) is approximated as

$$x_1^{(k+1)}(t) = x_1^{(0)}(t) + \int_\circ^t \lambda_1 \left[x'^{(k)}_1(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T)\right), \cdots, \tilde{x}_n^{(k)}(T) - g_1(T)\right] dT$$

$$\vdots \tag{3.4}$$

$$x_n^{(k+1)}(t) = x_n^{(0)}(t) + \int_\circ^t \lambda_n \left[x'^{(k)}_n(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T)\right), \cdots, \tilde{x}_n^{(k)}(T) - g_n(T)\right] dT,$$

where $\lambda_i = \pm 1, i = 1,2,3,\cdots,n$ are Langrange multipliers.

$\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \cdots, \tilde{x}_n^{(k)}$ denote the restricted variations. For $\lambda_i = 1, i = 1,2,3,\cdots,n$; we have the following iterative schemes:

$$x_1^{(k+1)}(t) = x_1^{(0)}(t) + \int_\circ^t \left[x'^{(k)}_1(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T)\right), \cdots, \tilde{x}_n^{(k)}(T) - g_1(T)\right] dT$$

$$\vdots \tag{3.5}$$

$$x_n^{(k+1)}(t) = x_n^{(0)}(t) + \int_\circ^t \left[x'^{(k)}_n(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T)\right), \cdots, \tilde{x}_n^{(k)}(T) - g_n(T)\right] dT.$$

The approximations is completely determined, if we start with the initial approximation $x_i(0) = c_i, i = 1,2,\cdots,n$. Finally, the solution

$$x_i(t) = \lim_{n\to\infty} x_i^{(n)}(t) \tag{3.6}$$

is approximated by the nth term $x_i^{(n)}(t), i = 1,2,3,\cdots,n$.

4. Tau-Reduction

We considered the linear boundary value problem for the nth order integro differential equation of the form:

$$y^n(x) = g(x) + f(x)y(x) + \lambda \int_\circ^x P(x,t)y(t)dt, \tag{4.1}$$

with initial conditions

$$y(0) = \alpha_\circ, y'(0) = \alpha_1, \cdots, y^{(n-1)}(0) = \alpha_{n-1}. \tag{4.2}$$

We considered the transformation

$$y(x) = y_1(x), \frac{dy}{dx} = y_2(x), \cdots, \frac{d^{(n-1)}y}{dx^{(m-1)}} = y_n(x) \tag{4.3}$$

and rewrite the above higher order boundary value problem as a system of differential equations:

$$\frac{dy_1}{dx} = y_2(x)$$

$$\frac{dy_2}{dx} = y_3(x) \tag{4.4}$$

$$\vdots$$

$$\frac{dy_n}{dx} = g(x) + f(x)y_1(x) + \lambda \int_\circ^x P(x,t)y_1(t)dt,$$

with initial conditions

$$y_1^{(0)}(x) = \alpha_\circ, y_2^{(0)}(x) = \alpha_1, \cdots, y_n^{(0)}(x) = \alpha_{n-1}.$$

The above system of differential equations are then perturbed and written as a system of integral equations with langrange multipliers $\lambda_i = 1, i = 1,2,3, \cdots, n$ as follows

$$y_1^{(p+1)}(x) = y_1^{(0)}(x) + \int_\circ^x y_2^{(p)}(s)ds + \tau_1 T_1(x)$$

$$y_2^{(p+1)}(x) = y_2^{(0)}(x) + \int_\circ^x y_3^{(p)}(s)ds + \tau_2 T_2(x) \tag{4.5}$$

$$\vdots$$

$$y_n^{(p+1)}(x) = y_n^{(0)}(x) + \int_\circ^x \left[g(s) + f(s)y_1^{(p)}(s) + \lambda \int_\circ^x p(s,t)y_1^{(p)}(t)dt \right] ds \quad + \tau_n T_n(x).$$

For example,

with $p = 0$, we obtain

$$y_1^{(1)}(x) = y_1^{(0)}(x) + \int_\circ^x \alpha_1 ds + \tau_1 T_1(x) = \alpha_\circ + \alpha_1 x + \tau_1 T_1(x)$$

$$y_2^{(1)}(x) = \alpha_1 + \int_\circ^x \alpha_2 ds + \tau_2 T_2(x) = \alpha_1 + \alpha_2 x + \tau_2 T_2(x) \qquad (4.6)$$

$$\vdots$$

$$y_n^{(1)}(x) = \alpha_{n-1} + \int_\circ^x \left[g(s) + f(s)\alpha_\circ + \lambda \int_\circ^x p(s,t)\alpha_\circ dt \right] ds + \tau_n T_n(x),$$

where $\tau_i, i = 1,2,\cdots,n$ are free tau parameters to be determined and $T_n(x)$ are Chebyshev Polynomials of degree n of the first kind which is valid in the interval $-1 \leq x \leq 1$ and is given by

$$T_n(x) = \cos(n\cos^{-1}x) \qquad (4.7)$$

and which satisfy the recurrence relation given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n \geq 1. \qquad (4.8)$$

5. Illustration of the Technique

Example 1:

Consider the linear fourth order integro-differential equation (He, 2007)

$$y^{(iv)}(x) = x(1 + e^x) + 3e^x + y(x) - \int_\circ^x y(t)dt \qquad (5.1)$$

$$y(0) = 1, y(1) = 1 + e, y'(0) = 1, y''(1) = 3e.$$

The exact solution for this problem is $y(x) = 1 + xe^x$.

Using the transformation $y_1 = y, y_2 = y', y_3 = y''$ and $y_4 = y'''$. We rewrite the above problem as a system of differential equations.

$$\frac{dy_1}{dx} = y_2(x)$$

$$\frac{dy_2}{dx} = y_3(x)$$

$$\frac{dy_3}{dx} = y_4(x) \tag{5.2}$$

$$\frac{dy_4}{dx} = x(1+e^x) + 3e^x + y_1(x) - \int_{\circ}^{x} y(t)dt.$$

The above system of differential equations are written as integral equations with Lagrange multiplier $\lambda_i = 1, i = 1,2,3,\cdots,n$

$$y_1^{(p+1)}(x) = y_1^{(0)}(x) + \int_{\circ}^{x} y_2^{(p)}(s)ds + \tau_1 T_1(x)$$

$$y_2^{(p+1)}(x) = y_2^{(0)}(x) + \int_{\circ}^{x} y_3^{(p)}(s)ds + \tau_2 T_2(x) \tag{5.3}$$

$$y_3^{(p+1)}(x) = y_3^{(0)}(x) + \int_{\circ}^{x} y_4^{(p)}(s)ds + \tau_3 T_3(x)$$

$$y_4^{(p+1)}(x) = y_4^{(0)}(x) + \int_{\circ}^{x} \left[s(1+e^s) + 3e^s + y_1^{(p)}(s) - \int_{\circ}^{s} y_1^{(s)}(t)dt\right] ds$$

$$+\tau_4 T_4(x)$$

with $y_1^{(0)} = 1, y_2^{(0)} = 1, y_3^{(0)} = A, y_4^{(0)} = B$.

Consequently, we obtained the following approximations:

1st iteration (i.e p=0)

$$y_1^{(1)} = 1 + x + \tau_1(2x-1)$$

$$y_2^{(1)} = 1 + (A - 8\tau_2)x + 8\tau_2 x^2 + \tau_2$$

$$y_3^{(1)} = A + (B + 18\tau_3)x - 48x^2\tau_3 + 32x^3\tau_3 - \tau_3$$

$$y_4^{(1)} = B + xe^x + 2e^x + x - 2 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)\tau_4.$$

2nd iteration (i.e p=1)

$$y_1^{(2)} = 1 + (1+\tau_2)x + \left(\frac{A}{2} - 4\tau_2\right)x^2 + \frac{8}{3}\tau_2 x^3$$

$$y_2^{(2)} = 1 + (A - \tau_3)x + (B + 18\tau_3)\frac{x^2}{2} - 16\tau_3 x^3 + 8\tau_3 x^4$$

$$y_3^{(2)} = A + (B + 2 + \tau_4)x + \left(\frac{1}{2} - 16\tau_4\right)x^2 + \frac{160}{3}\tau_4 x^3 - 64\tau_4 x^4$$

$$+\frac{128}{5}\tau_4 x^5 + xe^x + 2e^x - 1$$

$$y_4^{(2)} = B + (1 - \tau_1)x + (1 + 3\tau_1)\frac{x^2}{2} - (1 + 2\tau_1)\frac{x^3}{6} + xe^x + 2e^x - 2.$$

3rd iteration (i.e p=2)

$$y_1^{(3)} = 1 + x + (A - \tau_3)\frac{x^2}{2} + (B + 18\tau_3)\frac{x^3}{6} - 4\tau_3 x^4 + \frac{8}{5}\tau_3 x^5$$

$$y_2^{(3)} = 1 + Ax + (B + 2 + \tau_4)\frac{x^2}{2} + \left(\frac{1}{2} - 16\tau_4\right)\frac{x^3}{3} + \frac{40}{3}\tau_4 x^4 - \frac{64}{5}\tau_4 x^5 + \frac{64}{15}\tau_4 x^6 - xe^x - x$$

$$y_3^{(3)} = A + Bx + (1 - \tau_1)\frac{x^2}{2} + (1 + 3\tau_1\frac{x^3}{6} - (1 - 2\tau_1)\frac{x^4}{24} + xe^x + e^x - 2x - 1$$

$$y_4^{(3)} = B + xe^x + 2e^x + x + (1 + \tau_2)\frac{x^2}{2} + (A - 9\tau_2 - 1)\frac{x^3}{6} + \left(\tau_2 - \frac{A}{24}\right)\frac{x^4}{12} - \frac{2}{15}\tau_2 x^5 - 2.$$

4th iteration (i.e p=3)

$$y_1^{(4)} = 1 + x + A\frac{x^2}{2} + (B - 2 + \tau_4)\frac{x^3}{6} + \left(\frac{1}{2} - 16\tau_4\right)\frac{x^4}{12} + \frac{8}{3}\tau_4 x^5 - \frac{32}{15}\tau_4 x^6 + \frac{64}{105}\tau_4 x^7 - xe^x - e^x - \frac{x^2}{2} + 1$$

$$y_2^{(4)} = 1 + (A - 1)x + \left(\frac{B}{2} - 1\right)x^2 + (1 - \tau_1)\frac{x^3}{6} + (1 + 3\tau_1)\frac{x^4}{24} - (1 + 2\tau_1)\frac{x^5}{120} + xe^x$$

$$y_3^{(4)} = 1 + (B - 2)x + xe^x + e^x + \frac{x^2}{2} + (1 + \tau_1)\frac{x^3}{6} + (A - 9\tau_1 - 1)\frac{x^4}{24} + \left(\tau_2 - \frac{A}{24}\right)\frac{x^5}{60} - \frac{1}{45}\tau_2 x^6 - 2x - 1$$

$$y_4^{(4)} = B + xe^x + 2e^x + x + \frac{x^2}{2} + (A - \tau_3 - 1)\frac{x^3}{6} + (B - A + 19\tau_3)\frac{x^4}{24} - (B + 114\tau_3)\frac{x^5}{120} + \frac{2}{5}\tau_3 x^6 - \frac{4}{105}\tau_3 x^7 - 2.$$

Thus, using the initial boundary conditions, we obtained the values of $\tau_1(i = 1,2,3,4)$, A and B.

Example 2:

Consider the nonlinear fourth-order integro-differential equation (He, 2007)

$$y^{(iv)}(x) = 1 + \int_{\circ}^{x} e^{-x} y^2(t)dt,$$

with the boundary conditions

$$y(0) = 1, y'(0) = 1, y(1) = e, y''(1) = e.$$

The exact solution of the problem is $y(\mathrm{x}) = e^x$.

Using the transformation

$$y_1 = y, y_2 = y', y_3 = y'' \text{and } \ y_4 = y'''.$$

We rewrite the above problem as a system of differential equations.

$$\frac{dy_1}{dx} = y_2(x)$$

$$\frac{dy_2}{dx} = y_3(x)$$

$$\frac{dy_3}{dx} = y_4(x)$$

$$\frac{dy_4}{dx} = 1 + \int_{\circ}^{x} e^{-x} y_1^2(t)dt.$$

The above system of differential equations are written as integral equations with Lagrange multiplier $\lambda_i = 1, i = 1,2,3,\cdots,n$

$$y_1^{(p+1)}(x) = y_1^{(0)}(x) + \int_{\circ}^{x} y_2^{(p)}(s)ds + \tau_1 T_1(x)$$

$$y_2^{(p+1)}(x) = y_2^{(0)}(x) + \int_{\circ}^{x} y_3^{(p)}(s)ds + \tau_2 T_2(x)$$

$$y_3^{(p+1)}(x) = y_3^{(0)}(x) + \int_{\circ}^{x} y_4^{(p)}(s)ds + \tau_3 T_3(x)$$

$$y_4^{(p+1)}(x) = y_4^{(0)}(x) + \int_{\circ}^{x} \left[1 + \int_{\circ}^{s} e^{-s}(y_1(t))^2 dt\right] ds + \tau_4 T_4(x)$$

with $y_1^{(0)} = 1, y_2^{(0)} = 1, y_3^{(0)} = A, y_4^{(0)} = B$.

Consequently, we obtained the following approximations:

1st iteration (i.e p=0)

$$y_1^{(1)} = 1 + (1 - 2\tau_1)x - \tau_1$$

$$y_2^{(1)} = 1 + (A - 8\tau_2)x + 8\tau_2 x^2 + \tau_2$$

$$y_3^{(1)} = A + (B + 18\tau_3)x - 48x^2\tau_3 + 32x^3\tau_3 - \tau_3$$

$$y_4^{(1)} = B + xe^x + 2e^x + x - 2 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)\tau_4.$$

Continuing in the same manner, we obtain

3rd iteration (i.e p=2)

$$y_1^{(3)} = 1 + x + (A - \tau_3)\frac{x^2}{2} + (B + 18\tau_3)\frac{x^3}{6} - 4\tau_3 x^4 + \frac{8}{5}\tau_3 x^5$$

$$y_2^{(3)} = 1 + Ax + (B - 2 + \tau_4)\frac{x^2}{2} + \left(\frac{1}{2} - 16\tau_4\right)\frac{x^3}{3} + \frac{40}{3}\tau_4 x^4 - \frac{64}{5}\tau_4 x^5 + \frac{64}{15}\tau_4 x^6 - xe^x - x$$

$$y_3^{(3)} = A + (B + 5 + 8\tau_1 + 5\tau_1^2)x + \frac{x^2}{2} + (11 + 26\tau_1 + 17\tau_1^2)xe^{-x} + (16 + 34\tau_1 + 22\tau_1^2)e^{-x} + (3 + 9\tau_1 + 6\tau_1^2)x^2e^{-x} + \frac{1}{3}(1 + 4\tau_1 + 4\tau_1^2)x^3e^{-x} - (16 + 34\tau_1 + 22\tau_2)$$

$$y_4^{(3)} = B + x - (6A^2 + 230A\tau_2 + 3026\tau_2^2 + 8A + 86\tau_2 + 5)xe^{-x} - (6A^2 + 230A\tau_2 + 3026\tau_2^2 + 8A + 86\tau_2 + 5)e^{-x} - (2 - 21\tau_2 + 4A + 1513\tau_2^2 + 179A\tau_2 + 3A^2)x^2e^{-x} + \left(\frac{1}{3} + 8\tau_2 + \frac{1513}{3}\tau_2^2 + \frac{115}{3}\tau_2 + A^2\right)x^3e^{-x}$$

$$- \left(4\tau_2 + \frac{115}{12}A\tau_2 + 126A\tau_2^2 + \frac{A}{4} + \frac{A^2}{4}\right)x^4e^{-x} - \left(\frac{A^2}{20} + \frac{16}{15}\tau_2 + \frac{28}{15}A\tau_2 + \frac{128}{5}\tau_2^2\right)x^5e^{-x} - \left(\frac{4}{9}A\tau_2 + \frac{64}{18}\tau_2^2\right)x^6e^{-x} - \frac{64}{63}\tau_2^2x^7e^{-x} + (6A^2 + 230A\tau_2 + 3026\tau_2^2 + 8A + 86\tau_2 + 5).$$

Thus, using the boundary conditions, we obtained the values of $\tau_1 (i = 1,2,3,4)$, A and B.

Table 1: The results obtained by perturbed variational iterative method for example 1

x	Exact Sol	VIM	PVIM	Error (VIM)	Error(PVIM)
0.0	1.00000000	1.00000000	1.000000000	0.000000	0.000000
0.1	1.111051700	1.11105170	1.111052780	0.000000	1.006E-6
0.2	1.244280550	1.24428054	1.244281660	1.000E-8	1.080E-6
0.3	1.404957640	1.40495760	1.404960072	4.000E-8	1.110E-6
0.4	1.596729870	1.59672985	1.596732204	2.000E-8	2.430E-6
0.5	1.824360630	1.82436060	1.824362957	3.000E-8	2.330E-6
0.6	2.093271280	2.09327006	2.093273599	1.200E-6	2.327E-6
0.7	2.409622680	2.40962585	2.409629183	1.040E-6	2.319E-6
0.8	2.780432740	2.78043070	2.780435450	2.040E-6	2.290E-6
0.9	3.213642800	3.21364261	3.213645069	1.900E-7	2.269E-6
1.0	3.718281820	3.71828180	3.718282619	2.000E-8	1.990E-7

Table 2: The results obtained by perturbed variational iterative method for example 2

x	Exact Sol	VIM	PVIM	Error (VIM)	Error(PVIM)
0.0	1.00000000	1.00000000	1.00000000	0.000000	0.000000
0.1	1.105170918	1.10515817	1.10515817	1.100E-8	7.000E-9
0.2	1.221402758	1.22140225	1.22140225	2.000E-8	2.000E-8
0.3	1.349858800	1.34985778	1.34985778	1.200E-6	1.020E-6
0.4	1.491824690	1.49182437	1.49182437	3.200E-7	3.200E-7
0.5	1.648721271	1.64872034	1.64872034	9.300E-7	9.300E-7

0.6	1.822118800	1.82211777	1.82211777	1.000E-6	1.030E-6
0.7	2.013752707	2.01375078	2.01375078	1.900E-6	1.920E-6
0.8	2.225540920	2.22554083	2.22554083	9.000E-8	9.000E-8
0.9	2.459603110	2.45960310	2.45960310	1.000E-8	1.000E-8
1.0	2.718281820	2.71828185	2.71828153	2.900E-7	2.900E-7

6. Conclusion

In example 1, the results obtained by Perturbed Variational Iteration Method are closed to the results obtained by the Conventional Variational Iteration Method. We also observed from example 2 that the results obtained are in close agreement with the Conventional Variational Iteration Method. The added advantages of our method are, it is simple, easy and is easily programmed.

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