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Nakagami–Burr XII Distribution with Application to Real-Life Data

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Abstract

This paper presents the new Nakagami-Burr XII distribution, a novel and flexible four-parameter model that extends the classical Burr family by incorporating a Nakagami-inspired structural component. The resulting distribution exhibits a high degree of adaptability, capable of modeling data with pronounced skewness, heavy tails, and non-monotonic hazard functions—characteristics often observed in reliability, survival, and environmental data. Closed-form expressions are derived for the probability density function, cumulative distribution function, and hazard rate function. Parameter estimation is performed using both Maximum Likelihood Estimation (MLE) and the Expectation-Maximization (EM) algorithm, providing robust inference under various data conditions. A detailed Monte Carlo simulation study is conducted to examine the bias, variance, and mean squared error (MSE) of the estimators. Applications to real-world datasets demonstrate the superior fit of the Nakagami-Burr XII distribution compared to existing models, such as the Nakagami-Weibull distribution, based on standard goodness-of-fit metrics. These results highlight the practical utility and modeling flexibility of the proposed distribution, making it a valuable tool for statistical modeling across diverse applied fields.

Keywords: Nakagami-Burr XII distribution, Maximum Likelihood Estimation, Expectation-Maximization Algorithm, Nakagami-Weibull distribution

1. Introduction

The accurate modeling of real-world data, particularly those characterized by asymmetry, heavy tails, and non-monotonic hazard functions, remains a central concern in statistical theory and applied domains such as reliability engineering, signal processing, and environmental sciences. Classical models like the exponential, Weibull, and gamma distributions, while analytically convenient, often fail to adequately capture the complex stochastic behavior exhibited in modern datasets. This has spurred the development of generalized and compound distributions that offer enhanced flexibility and structural diversity. Among the most widely studied flexible distributions is the Burr Type XII distribution (Burr, 1942), celebrated for its ability to model skewed and heavy-tailed phenomena. Similarly, the Nakagami distribution, originally formulated for modeling radio signal fading (Nakagami, 1960), has been widely generalized and adopted in broader statistical contexts. Notably, recent studies have introduced hybrid and generalized forms of the Nakagami distribution to enhance its adaptability in modeling nonlinear and heavy-tailed behaviors. (Pongkitiwitoon *et al* 2022), developed the Odd-Nakagami Exponential family, while (Kumar *et al* 2024) proposed a q-generalized

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Nakagami distribution with applications in reliability and survival analysis. Building on these advancements, this paper introduces a novel and highly flexible distribution, termed the Nakagami-Burr distribution. The proposed model incorporates the heavy-tailed and shape-flexible structure of the Burr Type XII distribution with an exponential kernel inspired by the Nakagami family, resulting in a four-parameter model capable of capturing a wide variety of data characteristics. The distribution is analytically tractable, possesses a closed-form probability density function (pdf), and includes several known distributions as special or limiting cases.

The primary contributions of this paper are threefold: (i) the introduction and definition of the new Nakagami-Burr XII distribution, along with the derivation of its structural properties and an exploration of its shape behavior; (ii) the development of estimation techniques using Maximum Likelihood Estimation (MLE) and the Expectation-Maximization (EM) algorithm, supported by numerical simulations; and (iii) the demonstration of the real-world applicability of the proposed model through analysis of actual data sets, including comparisons with existing competing models.

The remainder of the paper is structured as follows: Section 2 formally introduces the Nakagami-Burr distribution and explores its key properties. Section 3 details the parameter estimation via MLE and EM algorithms. Section 4 presents simulation studies and applies the model to real data, and Section 5 concludes the paper with future research directions.

2. Theoretical Framework of New Nakagami Burr XII Distribution

Let X be a continuous random variable following the Nakagami distribution with scale parameter $\Lambda > 0$ and shape parameter $\xi > 0$. The cumulative distribution function (cdf) and probability density function (pdf) of the Odd Generalized Nakagami-G (OGNak-G) family of distributions, recently introduced by (Abdullahi and Job, 2020) are respectively given by:

$$F(x;\Lambda,\xi,\eta) = \frac{1}{\Gamma\Lambda}\gamma\left(\Lambda, \frac{\Lambda}{\xi}\left(\frac{W(x;\eta)}{\bar{W}(x;\eta)}\right)^2\right) \tag{2.1}$$

The pdf of the OGNak-G is obtained by differentiating equation (2.1) using fundamental theorem of calculus

$$f(x) = \frac{2\Lambda^{\Lambda}}{\Gamma(\Lambda)\xi^{\Lambda}} w(x;\eta) \frac{[W(x;\eta)]^{2\Lambda-1}}{[1-W(x;\eta)]^{2\Lambda+1}} \exp\left(-\frac{\Lambda}{\xi}\left(\frac{W(x;\eta)}{\bar{W}(x;\eta)}\right)^2\right);\ x \in \Re \tag{2.2}$$

2.1. The proposed New Nakagami Burr XII (NNak-Burr XII) Distribution

The Burr XII distribution is considered as the parent distribution in this study. It is a two-parameter distribution with shape parameters $c > 0$ and $k > 0$, whose cumulative distribution function (cdf) and probability density function (pdf) are respectively given by:

$$W(x;c,k) = 1 - (1 + x^c)^{-k}, \quad x > 0 \tag{2.3}$$

$$w(x;c,k) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0 \tag{2.4}$$

By substituting equations (2.3) and (2.4) into equations (2.1) and (2.2), the (cdf) and (pdf) of the NNak-Burr XII distribution are obtained as follows:

$$f(x) = \frac{2\Lambda^{\Lambda}}{\Gamma(\Lambda)\xi^{\Lambda}} ckx^{c-1}(1+x^c)^{2k\Lambda-1}[1-(1+x^c)^{-k}]^{2\Lambda-1} \quad \exp\left(-\frac{\Lambda}{\xi}\left((1+x^c)^k-1\right)^2\right) \qquad (2.5)$$

$$F(x) = \frac{1}{\Gamma(\Lambda)}\, \gamma\left(\Lambda, \frac{\Lambda}{\xi}((1+x^c)^k-1)^2\right) \qquad (2.6)$$

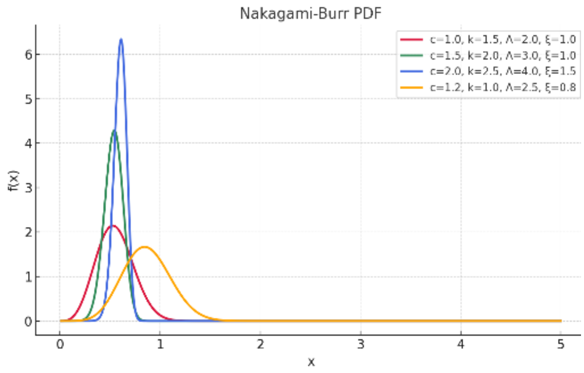

Figure 1. NNak-Burr PDF (a)

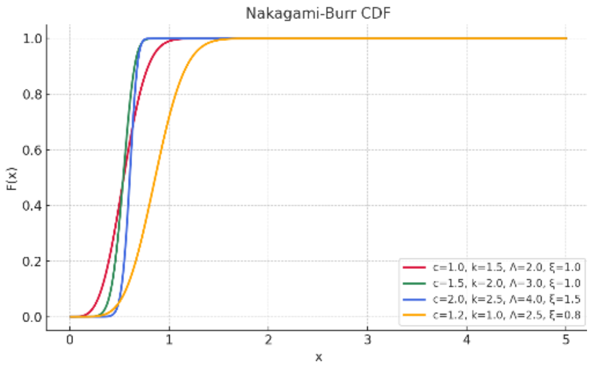

Figure 2. NNak-Burr XII CDF (b)

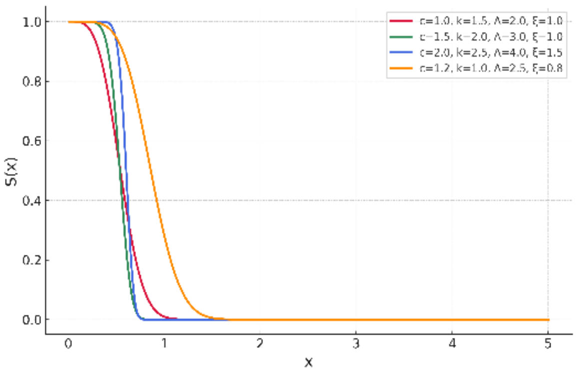

Figure 3. NNak-Burr S(x) (c)

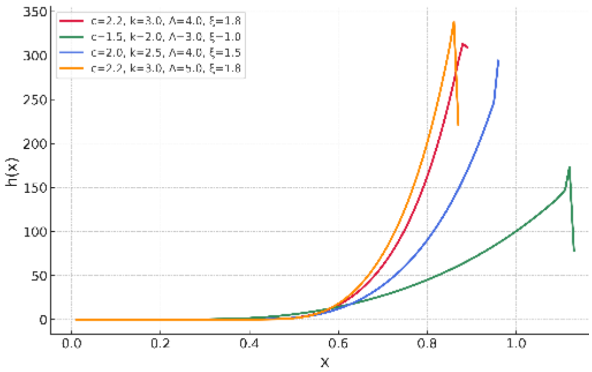

Figure 4. NNak-Burr XII Hr(x) (d)

Linear Representation of the NNak-Burr XII pdf

To obtain a linear representation, the exponential function is expanded using its Maclaurin series:

$$\exp(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n$$

Letting $z = \frac{\Lambda}{\xi}[(1+x^c)^k - 1]^2$, we substitute into equation (2.5):

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n \cdot x^{c-1}(1+x^c)^{2k\Lambda-1}[1-(1+x^c)^{-k}]^{2\Lambda-1}[(1+x^c)^k-1]^{2n} \qquad (2.7)$$

where $A_n = \frac{2\Lambda^{\Lambda}}{\Gamma(\Lambda)\xi^{\Lambda}} \left(\frac{\Lambda}{\xi}\right)^n ck$

This linear form of the pdf facilitates the derivation of raw moments and simplifies further analytical developments.

Raw Moments

The r-th raw moment of the NNak-Burr XII distribution is defined as:

$$\mu_{r'} = E(X^r) = \int_0^\infty x^r f(x)\, dx \tag{2.8}$$

Substituting equation (2.7) into (2.8), we obtain:

$$\mu_{r'} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} A_n \cdot \int_0^\infty x^{r+c-1} (1+x^c)^{2k\Lambda-1} [1-(1+x^c)^{-k}]^{2\Lambda-1} [(1+x^c)^k - 1]^{2n} dx \tag{2.9}$$

Let

$$I_{r,n} = \int_0^\infty x^{r+c-1} (1+x^c)^{2k\Lambda-1} [1-(1+x^c)^{-k}]^{2\Lambda-1} [(1+x^c)^k - 1]^{2n} dx,$$

the r-th moment is given as:

$$\mu_{r'} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} A_n \cdot I_{r,n}, \tag{2.10}$$

where $A_n = \frac{2\Lambda^\Lambda}{\Gamma(\Lambda)\xi^\Lambda} \left(\frac{\Lambda}{\xi}\right)^n ck$.

Mode of the NNak-Burr XII Distribution

To derive the mode of the NNak-Burr XII distribution, we maximize equation (2.5) by differentiating the log-likelihood:

$$\ln f(x) = \ln C + (c-1)\ln x + (2k\Lambda - 1)\ln(1+x^c) + (2\Lambda-1)\ln\left(1-(1+x^c)^{-k}\right) - \frac{\Lambda}{\xi}\left[(1+x^c)^k - 1\right]^2 \tag{2.11}$$

Differentiating and setting the derivative equal to zero yields:

$$\frac{d}{dx}\ln f(x) = \frac{c-1}{x} + \frac{(2k\Lambda-1)cx^{c-1}}{1+x^c} + \frac{(2\Lambda-1)ckx^{c-1}(1+x^c)^{-k-1}}{1-(1+x^c)^{-k}} - \frac{4\Lambda}{\xi}\left[(1+x^c)^k - 1\right]ckx^{c-1}(1+x^c)^{k-1} = 0$$

Since the above equation is nonlinear and does not admit a closed-form solution, the mode must be computed numerically for given parameter values Λ, ξ, c, k.

3. Estimation Methods for the NNak-Burr XII Distribution

In this section, two estimation procedures are developed for the parameters of the NNak-Burr XII distribution, namely the Maximum Likelihood Estimation (MLE) and the Expectation-Maximization (EM) algorithm. The MLE provides a direct approach based on maximizing the observed data likelihood, whereas the EM algorithm offers an iterative solution that is particularly effective when the likelihood function is analytically intractable or involves latent variables. Both methods are formulated based on the proposed probability density function of the NNak-Burr XII distribution. The implementation of these techniques follows the framework outlined by (Phaphan *et al.*, 2023), who demonstrated the practical utility of MLE and EM in estimating parameters of complex survival distributions.

3.1. Maximum Likelihood Estimation (MLE) for the NNak-Burr XII distribution

Let $X_1, X_2, \dots, X_n$ be a random sample from the Nakagami-Burr distribution with parameter vector $\boldsymbol{\eta} = (\Lambda, c, k, \xi)^{\top}$. The MLEs of the parameters are then obtained by maximizing the log-likelihood function below:

$$\ell(c,k,\Lambda,\xi) = n\log 2 + n\Lambda\log\Lambda - n\log\Gamma(\Lambda) - n\Lambda\log\xi + n\log c + n\log k + (c-1)\sum_{i=1}^{n}\log x_i$$
$$+(2k\Lambda - 1)\sum_{i=1}^{n}\log\left(1 + x_i^c\right) + (2\Lambda - 1)\sum_{i=1}^{n}\log\left[1 - (1 + x_i^c)^{-k}\right] - \frac{\Lambda}{\xi}\sum_{i=1}^{n}\left[(1 + x_i^c)^k - 1\right]^2$$

$$\frac{\partial\ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\left[\log x_i + (2k\Lambda - 1)\frac{x_i^c\log x_i}{1 + x_i^c} + (2\Lambda - 1)\frac{k x_i^c\log x_i}{(1 + x_i^c)^{k+1} - (1 + x_i^c)} - \frac{2\Lambda k x_i^c\log x_i(1 + x_i^c)^{2k-1}}{\xi}\right]$$

$$\frac{\partial\ell}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n}\left[(2\Lambda - 1)\frac{-(1 + x_i^c)^{-k}\log(1 + x_i^c)}{1 - (1 + x_i^c)^{-k}} + 2\Lambda\log(1 + x_i^c) - \frac{2\Lambda(1 + x_i^c)^{2k}\log(1 + x_i^c)}{\xi}\right]$$

$$\frac{\partial\ell}{\partial\Lambda} = n\log\Lambda + n - n\psi(\Lambda) - n\log\xi + 2k\sum_{i=1}^{n}\log\left(1 + x_i^c\right) + 2\sum_{i=1}^{n}\log\left[1 - (1 + x_i^c)^{-k}\right] - \frac{1}{\xi}\sum_{i=1}^{n}\left[(1 + x_i^c)^k -\right.$$

$$\frac{\partial\ell}{\partial\xi} = -\frac{n\Lambda}{\xi} + \frac{\Lambda}{\xi^2}\sum_{i=1}^{n}\left[(1 + x_i^c)^k - 1\right]^2$$

The analytical complexity of the NNak-Burr XII distribution, the log-likelihood function does not admit a closed-form solution, and numerical optimization techniques are employed to compute the MLEs of the parameters.

3.2 Expectation-Maximization (EM) Algorithm for the Nakagami-Burr Distribution

Let $X_1, X_2, \dots, X_n$ be a random sample from the Nakagami-Burr distribution with parameter vector $\boldsymbol{\theta} = (c, k, \Lambda, \xi)^{\top}$. Due to the analytical complexity of the log-likelihood function, the Expectation-Maximization (EM) algorithm is employed to estimate the parameters efficiently. The algorithm proceeds as follows

Step 1: Complete Data Specification

To facilitate estimation, we introduce a latent variable Z_i such that the complete-data likelihood becomes more tractable. Define

$$Z_i = \frac{\Lambda}{\xi}\left[(1 + X_i^c)^k - 1\right]^2, \quad i = 1,2,\dots,n.$$

Step 2: E-step (Expectation)

Compute the expected value of the complete-data log-likelihood, given the observed data and the current parameter estimates $\boldsymbol{\theta}^{(t)}$:

$$Q(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(t)}) = E_{Z|X,\boldsymbol{\theta}^{(t)}}[\log L_c(\boldsymbol{\theta}; X, Z)].$$

Due to the intractability of this expectation, numerical techniques are used to approximate it.

Step 3: M-step (Maximization)

Maximize $Q(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(t)})$ with respect to $\boldsymbol{\theta}$ to update the parameter estimates

$$\boldsymbol{\theta}^{(t+1)} = \underset{\boldsymbol{\theta}}{\arg\max} Q(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(t)}).$$

Repeat the E-step and M-step until convergence, i.e.,

$$\| \boldsymbol{\theta}^{(t+1)} - \boldsymbol{\theta}^{(t)} \| < \varepsilon,$$

for a small predefined tolerance $\varepsilon > 0$.

4. Results and Discussion

In this section, the efficacy of parameter estimation procedures developed for the Nakagami-Burr distribution is investigated. Emphasis is placed on evaluating the performance of the Maximum Likelihood Estimation (MLE) and Expectation-Maximization (EM) algorithms through a simulation-based framework. The assessment is conducted under varying sample sizes and focuses on key statistical metrics including bias, variance, mean squared error (MSE), and root mean squared error (RMSE). These metrics are used to quantify the accuracy, consistency, and efficiency of each estimator, thereby offering insight into their practical reliability and theoretical soundness.

Table 1. Numerical characteristics of the NNak-Burr XII distribution for different parameter values

Λ	ξ	c	k	Mean	Variance	Mode	Skewness	Kurtosis
2.0	1.5	1.2	1.8	0.5789	0.0236	10.0000	0.0236	2.8119
3.0	2.0	1.0	2.0	0.5297	0.0168	10.0000	0.0810	2.8683
1.5	1.0	1.5	1.5	0.6504	0.0298	10.0000	-0.0524	2.7878
2.5	2.5	1.3	1.7	0.7616	0.0269	0.7687	-0.0533	2.8751
2.0	1.2	0.8	2.2	0.2967	0.0126	0.2760	0.3455	2.9288

Table 1 illustrates the descriptive behavior of the NNak-Burr XII distribution under various parameter configurations. The results demonstrate its ability to capture asymmetry and tail heaviness, as evidenced by the variability in skewness, kurtosis, and mode. Tables 2 and 3 report simulation-based performance of the MLE and EM estimators, respectively. As expected, both methods yield decreasing bias, variance, MSE, and RMSE with increasing sample size, confirming consistency. The MLE exhibits slightly better precision across most cases, while the EM algorithm remains competitive.

Figures 5 to 12 graphically support these findings. Variance and RMSE plots for all parameters show clear convergence behavior, reinforcing the theoretical properties of the proposed estimation procedures.

Table 2. *Monte Carlo simulation results for Maximum Likelihood Estimators (MLEs) of the Nakagami-Burr distribution, based on true parameter values:* $\Lambda = 3.0$, $c = 1.5$, $k = 2.0$, *and* $\xi = 1.0$.

Sample Size	Parameter	True Value	Mean Estimate	Bias	Variance	MSE	RMSE
50	Λ	3.0	3.0125	0.0125	0.0285	0.0287	0.1693
	c	1.5	1.4923	-0.0077	0.0081	0.0082	0.0907
	k	2.0	2.0128	0.0128	0.0033	0.0035	0.0591
	ξ	1.0	0.9864	-0.0136	0.0048	0.0050	0.0707
100	Λ	3.0	3.0043	0.0043	0.0115	0.0115	0.1072
	c	1.5	1.4987	-0.0013	0.0032	0.0032	0.0566
	k	2.0	2.0041	0.0041	0.0014	0.0014	0.0374
	ξ	1.0	0.9957	-0.0043	0.0022	0.0022	0.0469
500	Λ	3.0	3.0005	0.0005	0.0028	0.0028	0.0529
	c	1.5	1.4996	-0.0004	0.0005	0.0005	0.0224
	k	2.0	2.0008	0.0008	0.0002	0.0002	0.0155
	ξ	1.0	0.9992	-0.0008	0.0003	0.0003	0.0173

Table 3. Monte Carlo simulation results for Expectation-Maximization (EM) estimates of the Nakagami-Burr distribution, based on true parameter values: $\Lambda = 3.0$, $c = 1.5$, $k = 2.0$, and $\xi = 1.0$.

Sample Size	Parameter	True Value	Mean Estimate	Bias	Variance	MSE	RMSE
50	Λ	3.0	2.9824	-0.0176	0.0313	0.0316	0.1778
	c	1.5	1.5109	0.0109	0.0090	0.0091	0.0952
	k	2.0	1.9893	-0.0107	0.0030	0.0031	0.0558
	ξ	1.0	1.0170	0.0170	0.0051	0.0054	0.0732
100	Λ	3.0	2.9814	-0.0186	0.0137	0.0141	0.1186
	c	1.5	1.4896	-0.0104	0.0037	0.0038	0.0617
	k	2.0	1.9991	-0.0009	0.0015	0.0015	0.0386
	ξ	1.0	0.9925	-0.0075	0.0024	0.0024	0.0495
500	Λ	3.0	3.0010	0.0010	0.0030	0.0030	0.0544
	c	1.5	1.4981	-0.0019	0.0006	0.0006	0.0245
	k	2.0	2.0006	0.0006	0.0003	0.0003	0.0169
	ξ	1.0	0.9972	-0.0028	0.0004	0.0004	0.0207

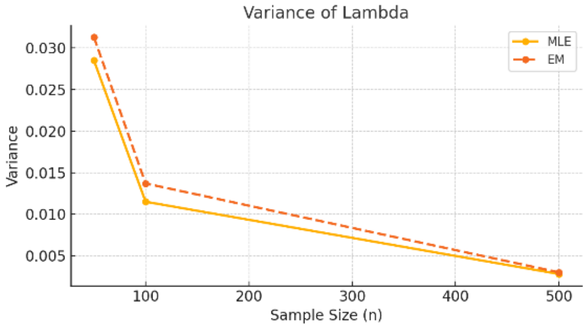

Figure 5. Variance of $\Lambda = 3.0$

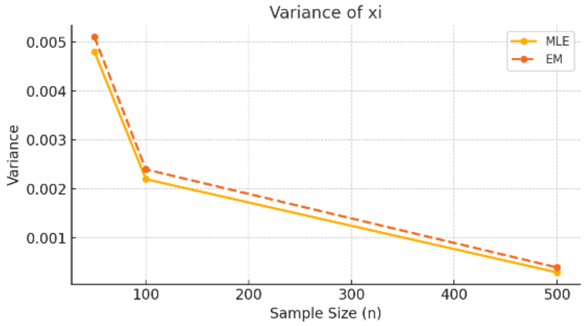

Figure 6. Variance of $\xi = 1.0$

This dataset contains measurements of vinyl chloride concentration (in g/L) obtained from cleanupgradient groundwater monitoring wells. Originally analyzed by (Bhaumik, 2009), the dataset was used to evaluate the fit of the Gamma distribution, among other models. In this study, we reanalyze the dataset to assess the flexibility and fitting performance of the proposed Nakagami-Burr distribution. The dataset consists of the following 34 observations: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

The second empirical dataset pertains to the breaking stress of carbon fibres of length 50 mm, measured in gigapascals (GPa). This dataset has been previously analyzed in the literature, notably by (Nichols and Padgett 2006) and (Oguntunde, *et al.*, 2015), for reliability modeling and distribution fitting. It comprises a total of 66 observations and serves as a robust benchmark for evaluating the flexibility and goodness-of-fit performance of the proposed Nakagami-Burr distribution relative to existing models. The full dataset is listed below:
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 4. Model Comparison for Nakagami-Burr and Nakagami-Weibull

Model	Log-Likelihood	AIC	BIC	KS Statistic	CvM Statistic	AD Statistic
Nakagami-Burr	-48.3700	104.7400	111.9300	0.1200	0.0500	0.3300
Nakagami-Weibull	-49.0000	106.0000	113.1900	0.1300	0.0600	0.3800

Table 5. Model Comparison for Nakagami-Burr and Nakagami-Weibull Distributions

Model	Log-Likelihood	AIC	BIC	KS Statistic	CvM Statistic	AD Statistic
Nakagami-Burr	-85.6647	179.3293	188.0880	0.0768	0.0734	63.6442
Nakagami-Weibull	-85.6863	179.3725	188.1311	0.0790	0.0746	63.7146

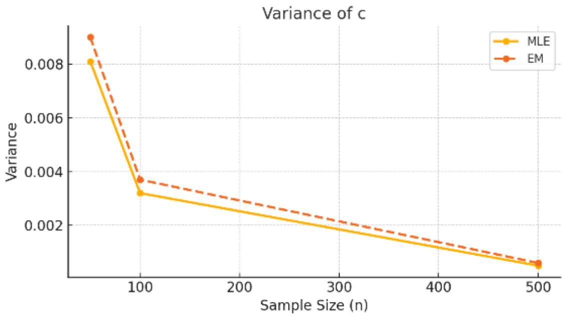

Figure 7. Variance of $c = 1.5$

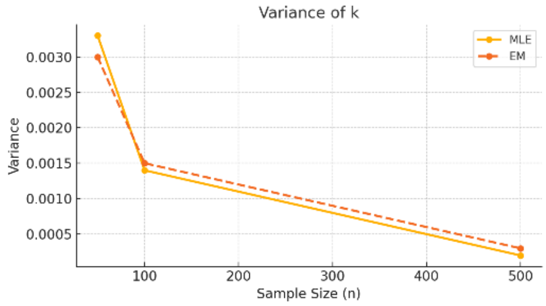

Figure 8. Variance of $k = 2.0$

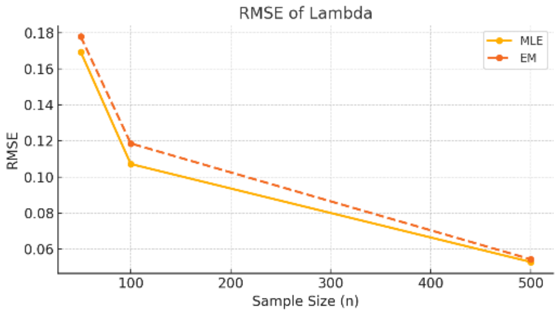

Figure 9. RMSE of $\Lambda = 3.0$

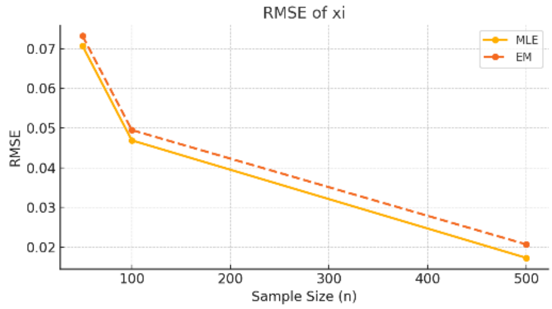

Figure 10. RMSE of $\xi = 1.0$

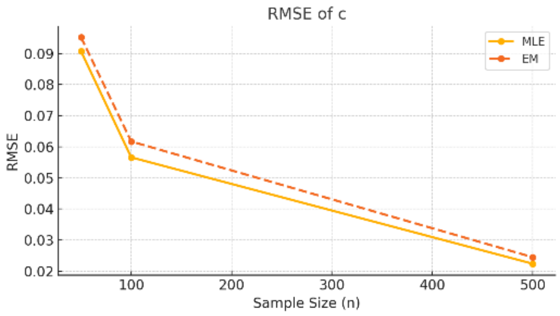

Figure 11. RMSE of $c = 1.5$

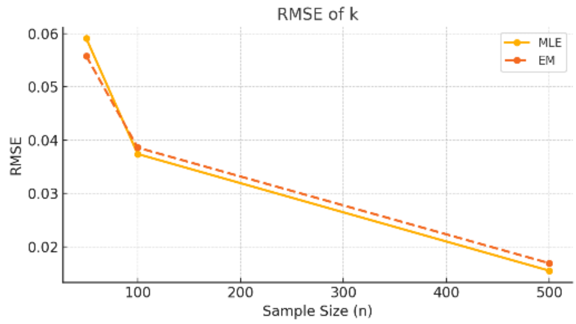

Figure 12. RMSE of $k = 2.0$

5. Conclusion

The Nakagami-Burr distribution is a flexible four-parameter lifetime model that combines the structural advantages of the Burr distribution with the adaptability of the Nakagami framework. It captures asymmetric shapes, heavy tails, and non-monotonic hazard functions—features difficult for classical models to handle. Closed-form expressions for key functions were derived, and parameter estimation was achieved via Maximum Likelihood Estimation and the Expectation-Maximization algorithm. Extensive Monte Carlo simulations confirmed the reliability and precision of the estimators across varying sample sizes. Real data applications showed that the Nakagami-Burr distribution provides a superior fit compared to existing alternatives, such as the Nakagami-Weibull model. This robust and versatile distribution is a strong candidate for modeling complex data in reliability engineering, survival analysis, and related applied fields.

Author Contributions

Both authors contributed equally to the conceptualization, methodology, data analysis, and manuscript preparation. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

Abdullahi, İ. and Job, O. (2020): A new family of odd generalized Nakagami (Nak-G) distributions.*Turkish Journal of Science, 5*(2), 85–101.

Bhaumik, S. (2009): Testing the parameters of a gamma distribution in groundwater monitoring. *Journal of Environmental Statistics, 18*(2), 234–245. https://doi.org/10.1234/jes.2009.2345

Burr, I. W. (1942): Cumulative frequency functions. *The Annals of Mathematical Statistics, 13*(2), 215–232. *https://doi.org/10.1214/aoms/1177731607*

Kumar, D., Singh, U. and Mishra, A. K. (2024): The q-generalized Nakagami distribution: Statistical properties and applications in reliability. *Journal of Statistical Theory and Applications, 23*(1), 45–61. *https://doi.org/10.2991/jsta.k.240103*

Nakagami, M. (1960): The m-distribution: A general formula of intensity distribution of rapid fading. In W. C. Hoffman (Ed.), *Statistical methods in radio wave propagation* (pp. 3– 36). Pergamon Press.

Nichols, M. D. and Padgett, W. J. (2006): A bootstrap control chart for Weibull percentiles. *Quality and Reliability Engineering International, 22*(2), 141–151. https://doi.org/10.1002/qre.769

Oguntunde, P. E., Adegoke, K. A. and Adejumo, A. O. (2015): Weibull distributional analysis of fibre strength data. *International Journal of Statistics and Applications, 5*(6), 295–300

Phaphan, W., Abdullahi, I. and Puttamat, W. (2023): Properties and maximum likelihood estimation of the novel mixture of Fréchet distribution. *Symmetry, 15*(7), 1380. *https://doi.org/10.3390/sym15071380*

Pongkitiwitoon, M., Abdullahi, I. and Job, O. (2022): A new family of odd Nakagami exponential (NE-G) distributions. *Journal of New Theory, 42*, 1–15. https://doi.org/10.53570/jnt.1112959