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Application of Lattice Theory on Order-Preserving Full Transformation Semigroup Via Fixed Points

Ibrahim, G. R.[1], Bakare, G. N.[2] , Akinwunmi. S. A.[3] and Aliu, O. T.[4]

[1,4]Department of Mathematics and Statistics, Kwara State University, Malete, P.M.B 1530, Ilorin, Kwara State, Nigeria
[2]Department of Mathematics, University of Ilorin, Ilorin. P.M.B, 1515 Ilorin, Kwara State, Nigeria.
[3]Department of Mathematics and Statistics, Federal University of Kashere, Gombe, Nigeria.

Abstract

Let X_n be a finite set, T_n be full transformation semigroup and OT_n be subsemigroup of all order-preserving full transformation semigroup. Let transformation $\alpha \in OT_n : (\forall x, y \in \text{Dom}\alpha)$, if $x \leq y$ then $\alpha(x) \leq \alpha(y)$, then α is called order-preserving transformation. This paper focuses on the notion of fixed points which are elements that remain unchanged under this transformation. That is, $\alpha(x) = x$ where α is a transformation on OT_n , x is the point in the $Dom(\alpha)$ and $\alpha(x)$ is the image of x , ∀x ∈ α The existence of fixed points was explored and emphasizing their role in establishing a lattice structure. The lattice of fixed points exhibits two essential operations: meet and join. These operations enable us to compute the greatest and least elements of fixed points. Beyond pure mathematics, the study of fixed points and their lattice structure has applications in dynamical systems, economics, computer science, and several other domains, making it both a theoretical and practical subject.

Keywords: Order-preserving full transformation, Lattice, Least elements, Greatest elements and Fixed Point.

1. Introduction

A lattice is a partially ordered set in which every two elements have a unique least upper bound and greatest lower bound. Least element in the lattice is an identity transformation, which fixes all elements in the set. Greatest element in the lattice corresponds to the transformation that maps all element to a single point. The meet of two transformations ω_1 and ω_2 is the transformation that fixes the element fixed by both ω_1 and ω_2 which is $\omega_1 \wedge \omega_2$. The join of two transformations ω_1 and ω_2 is the transformation that fixes the element fixed by at least one of ω_1 and ω_2 which is $\omega_1 \vee \omega_2$. Namboripad, (2005) studied on Lattice of Partial Transformation Semigroups, Laradji and Umar (2013) also work on Lattice Paths and Order-preserving Partial Transformation.

A Transformation semigroup is a pair(X,S), where X is a set and S is a semigroup of X. Here a transformation of X is just a function from a subset of X to X, not necessarily invertible, and

Corresponding Author: Baker, G.N.
Email: bakare.gn@unilorin.edu.ng

therefore S is simply a set of transformations of X which is closed under composition of functions. Let $X_n = \{1,2, \dots n\}$. The map $\alpha: Dom\alpha \subseteq X_n \rightarrow Im\alpha \subseteq X_n$ is said to befull or total, if Dom $\alpha = X_n$, partial if Dom $\alpha \subseteq X_n$ or else it is called strictly partial. The set of all partial transformation on n-object form a semi-group under the usual composition of transformation. Let T_n , P_n, and I_n be the full or total, partial and partial one-to-one on X_n respectively. These are the three essential part of transformation semigroups which were introduced by Howie (1995).

Order-preserving full transformation semigroups are algebraic structures that find applications in various mathematical and scientific domains. These semigroups consist of transformations on a set that not only preserve the underlying order but also form a semigroup under composition. The semigroups OT_n, are formed on a set S equipped with a partial order relation $\leq$. The binary operation in OT_n is typically composition. For $\omega, \delta \in OT_n$, the composition $\omega(\delta)$ is another order-preserving transformation. OT_n is closed under composition, which means that the composition of two order-preserving transformations results is another order-preserving transformation.

A fixed point of a transformation $\omega \in OT_n$ is a point x in the element ω such that $\omega(x) = x$. In other words, a fixed point is a point that remains unchanged under the action of the transformation. The study of fixed points in order-preserving full transformation semigroups reveals important insights into the behavior of these semigroups. One key property of order-preserving transformations is the existence of fixed points. For any order-preserving transformation ω, there exists at least one fixed point $x \in S$. This property is fundamental and underlies many applications of order-preserving transformations. Lattice theory offers a rich mathematical language and a set of tools to analyze the behaviour of order-preserving transformations and their fixed points. The meet and join operations in the lattice can be utilized to further study of fixed point. Some interesting combinatorial and algebraic results relating to semigroups of transformations have been investigated such as Laradji and Umar (2004) looked into Combinatorial Results for Semigroups of Order-preserving Partial Transformations, Bakare et al., (2014) worked on the number of order-preserving alternating semigroups, Ibrahim et al. **(**2019) also worked on some algebraic properties of order-preserving full contraction transformation. Recently, Laradji, (2022) established several results for the number $F(n,m)$ of elements of order-preserving partial transformation PO_n with m fixed points, including recurrence relations and generating functions. For standard concepts and terms in semigroup and lattice theory see [Dimitrova, and Koppitz, **(**2011) and Kemprast and Changhas, (2011)].

This paper shows that fixed points in order-preserving full transformation semigroups form a lattice structure.

2. Methods

The following Definitions and Examples were used to obtain our results.

Definition 2.1: Let ω be a transformation on OT_n, $\forall x, y \in \omega$ then $x \leq y$ implies $\omega(x) \leq \omega(y)$ is called order-preserving full transformations.

Definition 2.2: A fixed point is an element in the set that remains unchanged when the transformation is applied that is, $\omega(x) = x$ where ω is a transformation on OT_n , x is the point in the $Dom(\omega)$ and $\omega(x)$ is the image of x , $\forall x \in \omega$.

Definition 2.3: A lattice is a partially ordered set in which every two elements have a unique least upper bound and greatest lower bound. The following lattice properties can be used to analyze the fixed points.

Definition 2.4: The least element in the lattice represent the identity transformation, which fixes all element in the transformation.

Definition 2.5: The greatest element in the lattice corresponds to the transformation that maps all element to a single fixed point.

Definition 2.6: The meet of two transformations gives a transformation that fixes all elements fixed by both transformations.

Definition 2.7: The join of two transformations gives a transformation that fixes all elements fixed by at least one of the transformations.

Example 2.8: Consider transformations $\omega_1, \omega_2, \omega_3 \in OT_5$, such that
$\omega_1 = (1)(2)(3)(4)(5)$, $\omega_2 = (12](23](3)(4)(54]$, and $\omega_3 = (14](24](34](4)(54]$
Fixed points of ω_1: {1, 2, 3, 4, 5}
Fixed points of ω_2: {3, 4}
Fixed points of ω_3: {4}
Lattice properties:
Least element $= \omega_1$
greatest element $= \omega_3$
Meet of ω_1, ω_2 and ω_3: $\omega_1 \wedge \omega_2 \wedge \omega_3 = \omega_3$
Join of ω_1, ω_2 and ω_3: $\omega_1 \vee \omega_2 \vee \omega_3 = \omega_1$

Example 2.9: Consider transformations $\gamma_1, \gamma_2, \gamma_3 \in OT_{10}$ such that
γ_1= (1) (2)(3)(4)(5) (65] (76] (8910] (10)
γ_2 = (12] (2) (32] (42] (52] (62] (72] (82] (92] (102]
γ_3 = (1)(2)(3)(4)(5)(6)(7)(8)(9) (10)
Fixed points of γ_1: {1,2,3,4,5,10}
Fixed points of γ_2 :{2}
Fixed points of γ_3 : {1,2,3,4,5,6,7,8,9,10}
Lattice properties:
Least element = γ_3
greatest element = γ_2
Meet of γ_1, γ_2 *and* γ_3 : $\gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_2$
Join of γ_1, γ_2 *and* γ_3 : $\gamma_1 \vee \gamma_2 \vee \gamma_3 = \gamma_3$

Remarks: To show that elements of OT_n satisfy lattice properties, the following must hold:

(i) Identity element must exist.
(ii) There must be one element of height one that fixes all other points in the element.
(iii) The other elements must at least fix one point that is being fixed by (ii).

3. Results and Discussion

Lemma 3.1: Let $S = OT_n$, if there exists an identity element $\rho \in OT_n$, then

OT_n contains a least element.

Proof: Let ρ be the identity transformation on OT_n, such that$\rho = (1)\,(2)\,(3)\ldots(n)$. ρ fixes all elements, so it is order-preserving. For any order-preserving transformation ρ, we have$1.\rho = \rho.1 = \rho$, which means ρ is the identity element with respect to composition. Hence, ρ is the least element in the lattice.

Lemma 3.2: Let $S = OT_n$, for $\gamma \in OT_n$, where γ is a transformation of height one. Then, the greatest element exists in OT_n.

Proof:

Consider an element $\gamma \in OT_n$ such that, $\gamma = \begin{pmatrix} 1 & 2 & 3 & - & \sigma & - & - & n \\ & & & & \sigma & & & \end{pmatrix}$

Where σ is a single fixed point in γ and every point in the γ maps to σ, let's say x, $\gamma(x) = \sigma$, $\forall\, x \in \gamma$. Since γ preserve the order, it is order-preserving. Therefore, any order-preserving transformation S we have $\gamma.S = S.\gamma = \gamma$. Hence, γ is the greatest element in the lattice.

Theorem 3.3: Let $S = OT_n$, and $\omega, \gamma \in S$. Then ω and γ are closed under meet operation lattice properties.

Proof:

Given two transformations $\omega, \gamma \in OT_n$. Let $\omega \wedge \gamma$ denote the meet operation. The meet operation represents the composition of ω and γ that fixes the common fixed points of ω and γ. Let X be the set of elements fixed by both ω and γ. Then, $\omega * \gamma$ fixes all elements in X since composition preserves order and fixes the common fixed points. Thus, $\omega * \gamma$ is the meet of ω and γ.

Theorem 3.4.: Let $S = OT_n$, and $\omega, \gamma \in S$. Then ω and γ are closed under join lattice properties.

Proof: For two transformations $\omega, \gamma \in OT_n$. The join operation denoted as $\omega \vee \gamma$ is the operation represents the composition of ω and γ that fixes the union of the fixed points of ω and γ. Let Y be the set of elements fixed by at least one of ω and γ and Then, $\omega \vee \gamma$ fixes the union of fixed points. Thus, $\omega \vee \gamma$ is the join of ω and γ.

Theorem 3.5: Let $S = OT_n$, then S forms a lattice under composition.

Proof: To prove that elements of OT_n form a lattice, we need to show that it satisfies the defining properties of a lattice, which are the existence of a least element, a greatest element, and the closure under meet and join operations. These properties have been shown in lemmas 3.1, Theorem 3.2, 3.3, and 3.4 respectively. Hence, elements of OT_n form a lattice structure.

4. Conclusion

The lattice analysis helps us to identify the least element, greatest element, the meet and join operations on a given fixed point in OT_n. Fixed point representing the elements that remain unchanged under all transformations in the semigroup. This work shown that the least and greatest elements are identity and transformation of height one respectively. Also, the meet and join of given transformations are the union and intersection of their fixed points respectively. The existence of fixed points in OT_nwere explored in establishing lattice structure. Therefore, the application of lattice theory in the study of order-preserving full transformation semigroups provides a powerful framework for understanding the structure properties, and relationship within these semigroups

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