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Investigating Probability Distribution Associated with the Sum of at least Two Independent Positive Integers

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Abstract

This work examined the probabilities associated with the sum of at least two positive integers. Let Y equals the sum of independent positive integers be a random variable. The histograms, probability density function (pdf) and the cumulative probabilities of Y were examined. Besides, the skewness and kurtosis were estimated to be zero (0) and approximately three (3) respectively. The distribution of Y showed clearly bell-shaped curves. Furthermore, the parameters of normal variable Y for at least two positive integers exhibited linear relationship. With these accomplishments it becomes feasible to simulate non-negative normal data sets within certain range of observations available for utilization in further Statistical research intending to mimic real life problems and phenomena. The calculated probabilities, $P(Y = y_i)$ and cumulative probabilities, $P(Y \leq y_i)$ are valuables for inclusion in Standard Statistical Tables both for use by Students and Researchers.

Keywords: Integers, Probability generating function, Permutation, Combination, Simulation

1. Introduction

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Several authors have studied probability distributions of diverse events; viz events with only two possible outcomes of success or failure which follows binomial (n, p), repetition of an independent trial until a success occurs which is geometric (p) experiment, examination of the probability of

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rare events that follows Poison(λ) process, exponential(λ) distribution that monitors the decay or growth of a population, and normal random variables distributed with mean, μ_x, and variance σ_x^2, other distributions (Gnedenko, 1978: Grinstead and Snell, 1977: Murray and Spiegel, 1999; Maity, 2018; Turney, 2022a; Turney, 2022b and Bhandari, 2023). The calculation of probabilities typical of the sum of at least two independent positive integers will be explored using the concept of probability generating function (PGF). This holds only for positive integers whose sum is less than or equal to 9 yielding accurate estimates of the probabilities. On the contrary, using the concept of PGF to determine the probabilities of sums of positive integers, whose sums are greater than 9 breaks down as it offers conservative estimates of probabilities for example upward bias. This is one of the genuine motivations for this paper (Charles and Snell, 2012: Grinstead and Snell, 1977). This will be thoroughly investigated alongside with the empirical determination of the distribution that sums of independent positive integers follow.

It is very pertinent to reiterate that integers have wider applications in the day to day human activities in the following areas viz: banking industry, Firms, Companies, Ministries, Corporations just to mention but a few. The adoption of independent-digits positive integers enhances digitalization and injection of sanity (decorum), transparency, accountability, and security codes that guarantee smooth running of administration or electronic routine activities as the case may be. Specific examples include: the adoption of Automated Teller Machine (ATM) and Personal Identification Number (PIN) for ease of online financial transactions, issuance of bank accounts, bank verification number (BVN), car registration and use of plates number, students enrolments, statement of account, voters registration numbers, patterns of digits that abound in salaries of workforce across the tiers of government in Nigeria and world over among others. The relevancies of independent positive integers and its sum are not exhaustive. These have well informed motivations aimed at studying the possible numbers of permutations (with repetitions) or combinations or both, their sums, the associated probabilities, cumulative probabilities, pdf and the nature of parameters of distribution peculiar to the sum of independent positive integers.

2. Materials and Methods

The probability generation function (PGF) computational techniques are often used for studying integer valued discrete random variables. A detailed study of the conditions for the existence of PGF for discrete random variables allows the extension of the available methods for integers or real valued discrete random variables (Nakamura & Perez-Abreu,1993; Manuel, 2009)

Let PGF be denoted by

$$E(t^X) = P = _r(t) = p_0 t^0 + p_1 t + p_2 t^2 + p3t^3 + ... + p_i t^i = \sum_X P_X t^X \quad \text{for } t > 0 \qquad 1$$

$$\text{where } P_{r(j)}(t) = \frac{1}{i+1} \text{ for } j = 0,1,2,3,...,i \qquad 2$$

The natural domain of this PGF, $D_X = \{t > 0 : P_X(t) < +\infty\}$

Then from equation (1) $$P_r(t) = \left(\frac{1}{i+1}\right)(1+t+t^2+t^3+...+t^i) \quad 3$$

where $(1+t^1+t^2+...+t^i)$, is a finite geometric series with first term, a=1, common ratio, r=t and $n = 1+i$

The sum of a finite geometric series is given as, $S_{i+1} = \frac{a(1-r^n)}{(1-r)}, r < 1$ 4

This gives $p(t) = \left(\frac{1}{i+1}\right)(S_n) = \left(\frac{1}{n}\right)\left(\frac{(1-t^n)}{(1-t)}\right)$ 5

If k-independent positive integers are selected at random and we assume independence, we may wish to calculate the probability that the sum of the k-number of independent positive integers selected is a particular value $"c"$

$$P_r(t^{y1+y2+y3+...+yk}) = \left[\left(\frac{1}{n}\right)\left(\frac{(1-t^n)}{(1-t)}\right)\right]^k \quad 6$$

Since there are k positive integers being added together in (6)

$$p_r(t^{y1+y2+y3+...+yk}) = \frac{1}{n^k}\left[\frac{(1-t^n)}{(1-t)}\right]^k == \frac{1}{n^k}(1-t^n)^k(1-t)^{-k} \quad 7$$

The above involves the extension to the case of negative exponents of a binomial expression which resulted in double summation as below

$$p(t^{\sum_{i=1}^{k} y_i} = t^c) = \frac{1}{(n)^k}\sum_{i=0}^{k}\binom{k}{i}(-t)^{ni}\sum_{j=0}^{\infty}\binom{-k}{j}(-t)^j$$

$$\frac{1}{(n)^k}\sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\binom{-k}{j}(-1)^{ni+j}(t)^{ni+j} \quad 8$$

where $\binom{-k}{j} = \frac{(-k)(-k-1)(-k-2)....(-k-j+1)}{j!}$

The probability that the sum of the k-positive integers is "c" is the coefficient of $t^{ni+j} = t^c$

which is equal to $\frac{1}{n^k}\binom{k}{i}\binom{-k}{j}(-1)^j$ such that $i=0$ and $n=10$ as the number of positive integers. The PGF is suitable only for calculating probability of sums of k-positive integers that are less or equal to 9. No more, no less. The PGF breaks down to estimate probability accurately when the sum of positive integers is greater than 9, i.e. it gives conservative probabilities estimates. These are illustrated in Table 1.

Table 1: Table of probability

Number of positive integers	2	3	4	5	6	7
Using pgf , when sum, y=9, P(y=9)	0.10	0.55	0.2200	0.00715	0.002002	0.0005005
Using permutation or combination , when sum, y=9,P(y=9)	0.10	0.55	0.2200	0.00715	0.002002	0.0005005
Using pgf , when sum, y=10, P(y=10)	0.11	0.66	0.0286	0.01001	0.003003	0.0008008
Using permutation or combination, when sum, y=10,P(y=10)	0.09	0.63	0.0282	0.00996	0.002997	0.0008001

The alternative techniques involve the knowledge of permutations with replacement (repetitions), combinations or both as the case may be and theory of probability respectively.

Under Permutations with repetitions, independent draws are being made to yield permutations of multi set (a set of objects) some of which are alike. Whereas drawing at random without replacement enhances draws that are dependent using combinatoric technique (Seymour and Lipson, 2004)

Thus $p(n_1,n_2,n_3,...,n_r) = \frac{n!}{n_1!n_2!n_3!...n_r}$, where p stands for permutation

Then $p_r(y_i=n_1+n_2+n_3+...+n_r) = \frac{p(n_1,n_2,n_3,...,n_r)}{n(S)}$, where $n(S)=10^k$ for finding the probability that the sum of number of independent integers is a particular value "c"

For two (2) independent number of positive integers, using set notations we have

$$S_1 = \{(x_1, x_2)/0 \le x_1 \le 9, 0 \le x_2 \le 9, y = x_1 + x_2\}$$

For three (3) independent number of positive integers, using set notations we have

$$S_2 = \{(x_1, x_2, x_3)/0 \le x_1 \le 9, 0 \le x_2 \le 9, 0 \le x_3 \le 9, y = x_1 + x_2 + x_3\}$$

Therefore, in each case above, $P_r(y) = n(y)/S_j$

This study entails application of the knowledge of Permutations, Combinations or both in the calculation of the probabilities that the sum of the number of independent positive integers is a particular value using R-package in the procedures that follow:

PROCEDURES

Step

i. Select the number of independent positive integers (k), such that k=2,3, 4,..,10
ii. Select the values of positive integers as 0,1,2,3,4,5,6,7,8,9 respectively
iii. Obtain the permutations of n=10 elements in step (ii) with or without repetitions taking at least two (2) positive integers at a time.
iv. Let Y equals the sum of at least two (2) positive integers be a random variable, hence obtain the sum of elements in the permutations or combinatorial analysis or both in (iii)
v. Tabulate the frequency distribution for the r. v. Y in (iv)
vi. obtain the relative frequencies and cumulative relative frequencies for the r. v. Y as the probabilities, $P(Y = y_i)$ and cumulative probabilities, $P(Y \leq y_i)$ respectively

The art of selecting three positive integers for permutations or combinations with or without repetitions from 10 integers and implementing Steps (ii) through (vi) will be implemented using R-Package

Numerical Example1 for Illustration on Calculation of Probabilities for Sum of two (2) Positive Integers

The sums of the individual permutations or the mixed permutations and combinations as the random variable(Y) with associated frequencies are presented in the frequency Table 2, when the number of positive integers is 2

Table 2: Frequency distribution of the sum of two (2) positive integers

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
f	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1

Mean(Y) = 9, Median(Y) =9, Mode(Y) =9, Variance(Y) =16.6667, CV=0.4536
Skewness(Y) =0, Kurtosis(Y) =2.387879, Range=18

Numerical Example2 for Illustration on Calculation of Probabilities for Sum of three (3) Positive Integers

The sums of the permutations or the mixed permutations as well as combinations of three (3) digits integers as the random variable(Y) with their corresponding frequencies are presented in the frequency Table 3.

Table 3: Frequency distribution of the sum of three (3) digits positive integers

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75
y	15	16	17	18	19	20	21	22	23	24	25	26	27		
f	73	69	63	55	45	36	28	21	15	10	6	3	1		

Mean(Y) = 13.5, Median(Y) =13.5, Mode(Y) =13.5, Variance(Y) =24.7748, CV=0.3687
Skewness(Y) =0, Kurtosis(Y) =2.59192, Range=27

3. Results and Discussion

Table 4: Table of Frequency Distribution of Sum of at least Two Independent Positive Integers

Let k be the number independent positive integers

	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
y	f	f	f	f	f	f	f	f	f
0	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
2	3	6	10	15	21	28	36	45	55
3	4	10	20	35	56	84	120	165	220
4	5	15	35	70	126	210	330	495	715
5	6	21	56	126	252	462	792	1287	2002
6	7	28	84	210	462	924	1716	3003	5005
7	8	36	120	330	792	1716	3432	6435	11440
8	9	45	165	495	1287	3003	6435	12870	24310
9	10	55	220	715	2002	5005	11440	24310	48620
10	9	63	282	996	2997	8001	19440	43749	92368
11	8	69	348	1340	4332	12327	31760	75501	167860
12	7	73	415	1745	6062	18368	50100	125565	293380
13	6	75	480	2205	8232	26544	76560	202005	495220
14	5	75	540	2710	10872	37290	113640	315315	810040
15	4	73	592	3246	13992	51030	164208	478731	1287484
16	3	69	633	3795	17577	68145	231429	708444	1992925
17	2	63	660	4335	21582	88935	318648	1023660	3010150
18	1	55	670	4840	25927	113575	429220	1446445	4443725
19		45	660	5280	30492	142065	566280	2001285	6420700
20		36	633	5631	35127	174195	732474	2714319	9091270
21		28	592	5875	39662	209525	929672	3612231	12628000
22		21	540	6000	43917	247380	1158684	4720815	17223250
23		15	480	6000	47712	286860	1419000	6063255	23084500
24		10	415	5875	50877	326865	1708575	7658190	30427375
25		6	348	5631	53262	366135	2023680	9517662	39466306
26		3	282	5280	54747	403305	2358840	11645073	50402935
27		1	220	4840	55252	436975	2706880	14033305	63412580
28			165	4335	54747	465795	3059100	16663185	78629320
29			120	3795	53262	488565	3405600	19502505	96130540
30			84	3246	50877	504315	3735720	22505751	115921972
31			56	2710	47712	512365	4038560	25614639	137924380
32			35	2205	43917	512365	4303545	28759500	161963065
33			20	1745	39662	504315	4521000	31861500	187761310
34			10	1340	35127	488565	4682700	34835625	214938745
35			4	996	30492	465795	4782360	37594305	243015388
36			1	715	25927	436975	4816030	40051495	271421810
37				495	21582	403305	4782360	42126975	299515480
38				330	17577	366135	4682700	43750575	326602870
39				210	13992	326865	4521000	44865975	351966340
40				126	10872	286860	4303545	45433800	374894389
41				70	8232	247380	4038560	45433800	394713550

42	35	6062	209525	3735720	44865975	410820025
43	15	4332	174195	3405600	43750575	422709100
44	5	2997	142065	3059100	42126975	430000450
45	1	2002	113575	2706880	40051495	432457640
46		1287	88935	2358840	37594305	430000450
47		792	68145	2023680	34835625	422709100
48		462	51030	1708575	31861500	410820025
49		252	37290	1419000	28759500	394713550
50		126	26544	1158684	25614639	374894389
51		56	18368	929672	22505751	351966340
52		21	12327	732474	19502505	326602870
53		6	8001	566280	16663185	299515480
54		1	5005	429220	14033305	271421810
55			3003	318648	11645073	243015388
56			1716	231429	9517662	214938745
57			924	164208	7658190	187761310
58			462	113640	6063255	161963065
59			210	76560	4720815	137924380
60			84	50100	3612231	115921972
61			28	31760	2714319	96130540
62			7	19440	2001285	78629320
63			1	11440	1446445	63412580
64				6435	1023660	50402935
65				3432	708444	39466306
66				1716	478731	30427375
67				792	315315	23084500
68				330	202005	17223250
69				120	125565	12628000
70				36	75501	9091270
71				8	43749	6420700
72				1	24310	4443725
73					12870	3010150
74					6435	1992925
75					3003	1287484
76					1287	810040
77					495	495220
78					165	293380
79					45	167860
80					9	92368
81					1	48620
82						24310
83						11440
84						5005
85						2002
86						715
87						220
88						55
89						10
90						1

Table 5: Estimate of Parameters of Normal Distribution for The Sums of Independent Positive Integers

Number of positive integers(k)	2	3	4	5	6	7	8	9	10
N	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
Range	18	27	36	45	54	63	72	81	90
Mean(Y)	9	13.5	18	22.5	27	31.5	36	40.5	45
Median(Y)	9	13.5	18	22.5	27	31.5	36	40.5	45
Mode(Y)	9	13.5	18	22.5	27	31.5	36	40.5	45
Var(Y)	16.5	24.75	33.00	41.25	49.50	57.75	66	74.25	82.5
CV	0.4536	0.3687	0.3191	0.2854	0.2606	0.2412	0.2257	0.2128	0.2018
Skewness	0	0	0	0	0	0	0	0	0
Kurtosis	2.3879	2.5919	2.6939	2.7552	2.7960	2.8251	2.8470	2.8640	2.8776
Normal	N (9,16.5)	N(13.5,24.75)	N(18,33)	N(22.5,41.25)	N(27,49.50)	N(31.5,57.75)	N(36,66)	N(40.5,74.25)	N(45,82.5)

From the analysis, the means and variances of sum of independent positive integers grow linearly as the number of independent positive integers increase. These can be observed in the Figures 1 and 2 below

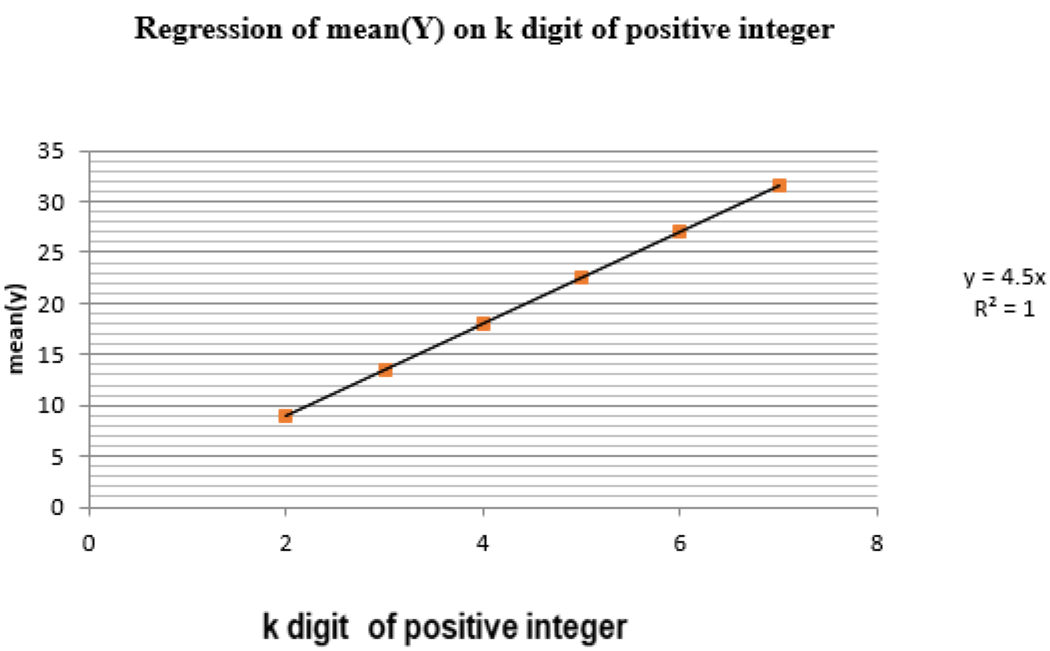

Figure1: Fitting mean(Y) on X
Footnote: X denotes k digit of positive integers, Y denotes the sum of k digit of positive integers

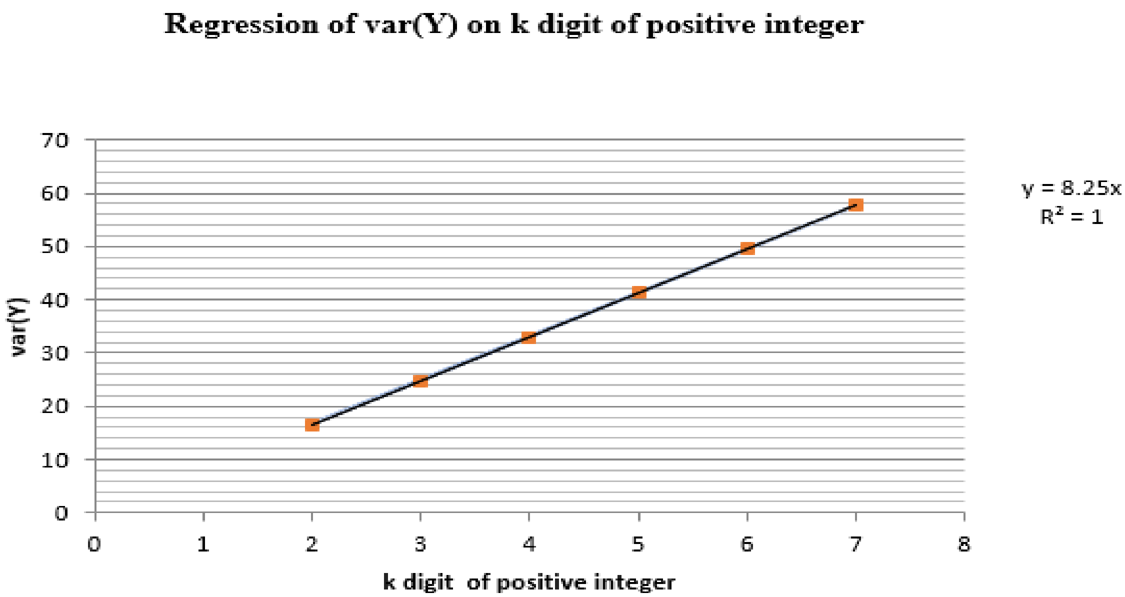

Figure2: Fitting var(Y) on X
Footnote: X denotes k digit of positive integers, Y denotes the sum of k digit of positive integers

The parameters of the normal distribution fitted to the random variable Y can be expressed in a general form as $\mu_{y_k} = 4.5*k$ And $\sigma^2_{y_k} = 0.4295 + 8.1921*k$, where k=2,3,4,5,6,7,8,9,10 denotes the number of independent positive integers . The normal random variable Y_k is given as $Y_k \sim N(4.5*k, 0.4295 + 8.1921*k)$

The Calculation of probabilities to 2 sig. figures for sum of independent positive integers 2, 3, 4... 7 using Continuity Correction for Normal Distribution Approximation to Binomial Distribution (Berry, 1841; Fischer, 1991; Pestman, 1988; Marques and Perez-Abreu, 1989; Keller and Warrack, 2003; Ramachandran and Chris, 2009; Sunklodas ,2012)

Table 6: Calculation of probabilities to 2 sig. figures for sum of independent positive integers 2, 3, 4... 7

NIPI	2	3	4	5	6	7
$E(Y)$	9	13.5	18	22.5	27	31.5
var(Y)	16.5	24.75	33	41.25	49.50	57.75
$\Pr(z \le y)$	$\Pr\left(z \le \frac{y+0.5-9}{4.0620}\right)$	$\Pr\left(z \le \frac{y+0.5-13.5}{4.9749}\right)$	$\Pr\left(z \le \frac{y+0.5-18}{5.7446}\right)$	$\Pr\left(z \le \frac{y+0.5-22.5}{6.4226}\right)$	$\Pr\left(z \le \frac{y+0.5-27}{7.0356}\right)$	$\Pr\left(z \le \frac{y+0.5-31.5}{7.5993}\right)$

Footnote: NIPI-Number of independent positive integers

Testing for the normality of the sums of at least two independent positive integers

Testing for normality was achieved by examination of histograms, construction of pdf and the Q-P plot in Figures 3, 4 and 5 respectively.

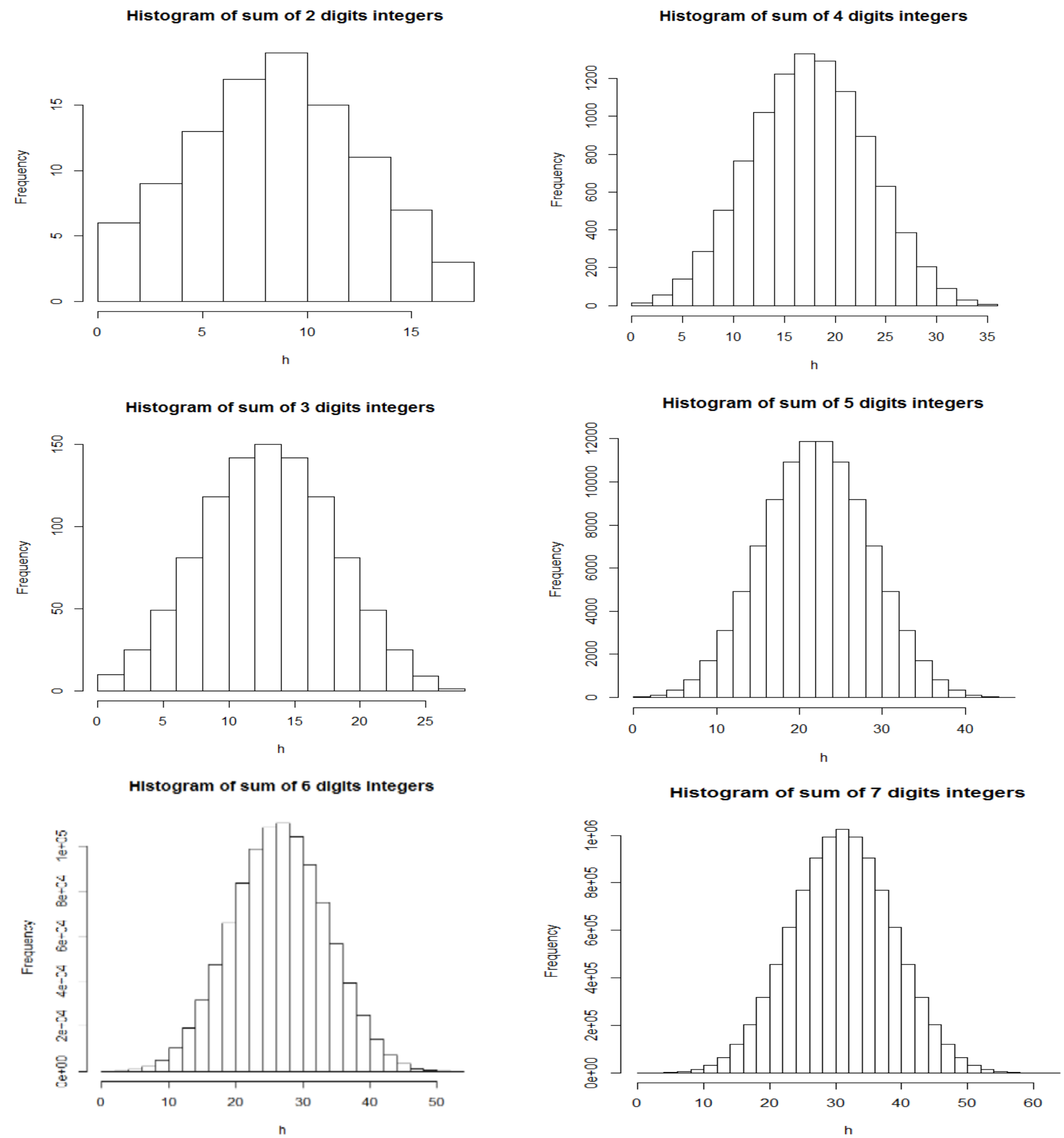

Figure3: Histograms for normality

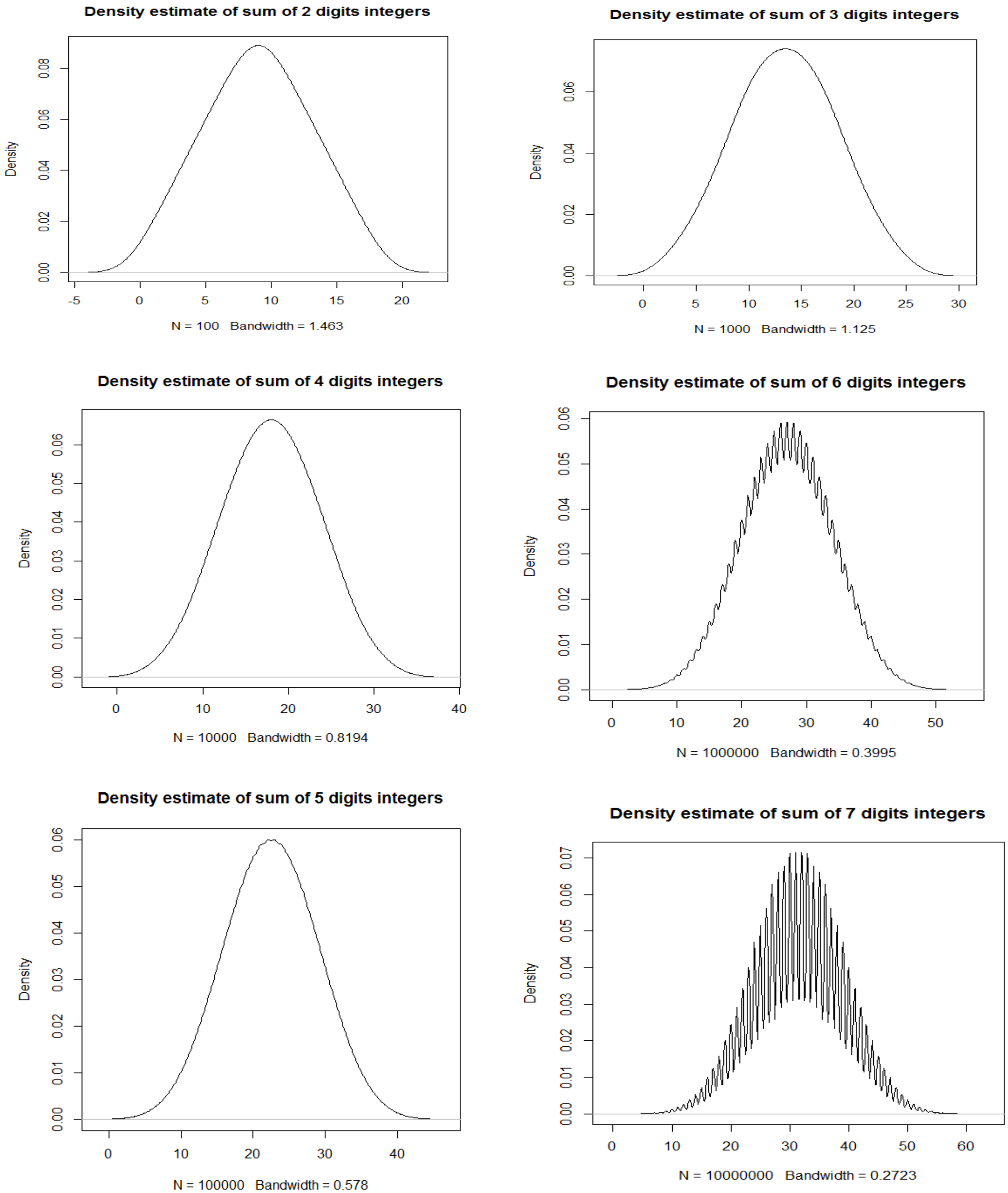

Figure 4: The Probability density function

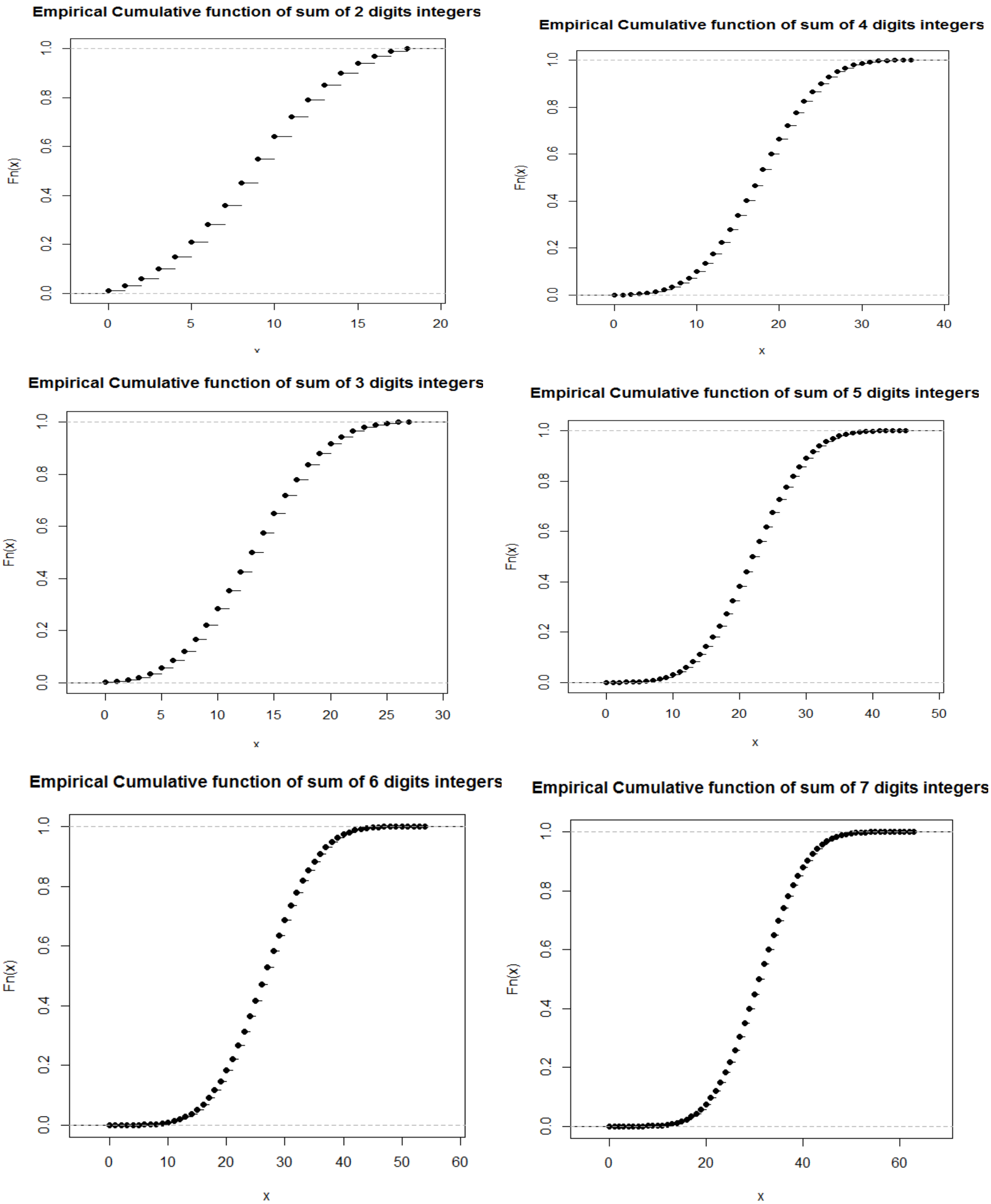

Figure 5: The relevant Q-P plot

4. Conclusion

From this study the frequency distribution of the sum of at least two independent positive integers are uni-modal for even number of independent positive integers and bimodal for odd number of positive integers. The mean, median and mode of the sum of independent positive integers are equal and this normality. The PGF breaks down when the sum of independent positive integers is greater than 9. The estimated skewness and kurtosis were 0 and 3 respectively corroborating that the r. v. Y is normally distributed. In addition, the histograms, pdf and normal probability plots for the frequency distribution of Y displayed features of normal distribution. The marginal propensity to increase in the means and variances of the sum of independent positive integers are 4.5 and 8.1921 respectively (Gaenssler, 1938). These are summarized in the normality distributional properties of Y. Therefore, this distributional property can be used in the simulation of non-negative normal data sets and Table of random numbers. Finally, the probabilities and cumulative probabilities for the sums of k-positive integers can be obtained (see Appendix "A1" & "A2") for inclusion in any Statistical Table for pedagogical, Students use and applications to mimic real life problems and phenomena.

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APPENDIX “A1”

Table 7: Estimate of Probability of sum of independent positive integers, k=2,3, 4,…,10

	k=2	k=3	k=4	k=5	k=6	k=7	**k=8**	**k=9**	**k=10**
y	$p(y_{k=2})$	$p(y_{k=3})$	$p(y_{k=4})$	$p(y_{k=5})$	$p(y_{k=6})$	$p(y_{k=7})$	$p(y_{k=8})$	$p(y_{k=9})$	$p(y_{k=10})$
0	**0.01**	**0.001**	**0.0001**	**0.00001**	**0.000001**	**0.0000001**	**0.00000001**	**0.000000001**	**0.0000000001**
1	**0.02**	**0.003**	**0.0004**	**0.00005**	**0.000006**	**0.0000007**	**0.00000008**	**0.000000009**	**0.0000000010**
2	**0.03**	**0.006**	**0.0010**	**0.00015**	**0.000021**	**0.0000028**	**0.00000036**	**0.000000045**	**0.0000000055**
3	**0.04**	**0.010**	**0.0020**	**0.00035**	**0.000056**	**0.0000084**	**0.00000120**	**0.000000165**	**0.0000000220**
4	**0.05**	**0.015**	**0.0035**	**0.00070**	**0.000126**	**0.0000210**	**0.00000330**	**0.000000495**	**0.0000000715**
5	**0.06**	**0.021**	**0.0056**	**0.00126**	**0.000252**	**0.0000462**	**0.00000792**	**0.000001287**	**0.0000002002**
6	**0.07**	**0.028**	**0.0084**	**0.00210**	**0.000462**	**0.0000924**	**0.00001716**	**0.000003003**	**0.0000005005**
7	**0.08**	**0.036**	**0.0120**	**0.00330**	**0.000792**	**0.0001716**	**0.00003432**	**0.000006435**	**0.0000011440**
8	**0.09**	**0.045**	**0.0165**	**0.00495**	**0.001287**	**0.0003003**	**0.00006435**	**0.000012870**	**0.0000024310**
9	**0.10**	**0.055**	**0.0220**	**0.00715**	**0.002002**	**0.0005005**	**0.00011440**	**0.000024310**	**0.0000048620**
10	**0.09**	**0.063**	**0.0282**	**0.00996**	**0.002997**	**0.0008001**	**0.00019440**	**0.000043749**	**0.0000092368**
11	**0.08**	**0.069**	**0.0348**	**0.01340**	**0.004332**	**0.0012327**	**0.00031760**	**0.000075501**	**0.0000167860**
12	**0.07**	**0.073**	**0.0415**	**0.01745**	**0.006062**	**0.0018368**	**0.00050100**	**0.000125565**	**0.0000293380**
13	**0.06**	**0.075**	**0.0480**	**0.02205**	**0.008232**	**0.0026544**	**0.00076560**	**0.000202005**	**0.0000495220**
14	**0.05**	**0.075**	**0.0540**	**0.02710**	**0.010872**	**0.0037290**	**0.00113640**	**0.000315315**	**0.0000810040**
15	**0.04**	**0.073**	**0.0592**	**0.03246**	**0.013992**	**0.0051030**	**0.00164208**	**0.000478731**	**0.0001287484**
16	**0.03**	**0.069**	**0.0633**	**0.03795**	**0.017577**	**0.0068145**	**0.00231429**	**0.000708444**	**0.0001992925**
17	**0.02**	**0.063**	**0.0660**	**0.04335**	**0.021582**	**0.0088935**	**0.03186480**	**0.001023660**	**0.0003010150**
18	**0.01**	**0.055**	**0.0670**	**0.04840**	**0.025927**	**0.0113575**	**0.00429220**	**0.001446445**	**0.0004443725**
19		**0.045**	**0.0660**	**0.05280**	**0.030492**	**0.0142065**	**0.00566280**	**0.002001285**	**0.0006420700**
20		**0.036**	**0.0633**	**0.05631**	**0.035127**	**0.0174195**	**0.00732474**	**0.002714319**	**0.0009091270**
21		**0.028**	**0.0592**	**0.05875**	**0.039662**	**0.0209525**	**0.00929672**	**0.003612231**	**0.0012628000**
22		**0.021**	**0.0540**	**0.06000**	**0.043917**	**0.0247380**	**0.01158684**	**0.004720815**	**0.0017223250**
23		**0.015**	**0.0480**	**0.06000**	**0.047712**	**0.0286860**	**0.01419000**	**0.006063255**	**0.0023084500**
24		**0.010**	**0.0415**	**0.05875**	**0.050877**	**0.0326865**	**0.01708575**	**0.007658190**	**0.0030427375**
25		**0.006**	**0.0348**	**0.05631**	**0.053262**	**0.0366135**	**0.02023680**	**0.009517662**	**0.0039466306**
26		**0.003**	**0.0282**	**0.05280**	**0.054747**	**0.0403305**	**0.02358840**	**0.011645073**	**0.0050402935**
27		**0.001**	**0.0220**	**0.04840**	**0.055252**	**0.0436975**	**0.02706880**	**0.011403331**	**0.0063412580**
28			**0.0165**	**0.04335**	**0.054747**	**0.0465795**	**0.03059100**	**0.016663185**	**0.0078629320**
29			**0.0120**	**0.03795**	**0.053262**	**0.0488565**	**0.03405600**	**0.019502505**	**0.0096130540**
30			**0.0084**	**0.03246**	**0.050877**	**0.0504315**	**0.03735720**	**0.022505751**	**0.0115921972**
31			**0.0056**	**0.02710**	**0.047712**	**0.0512365**	**0.04038560**	**0.025614639**	**0.0137924380**
32			**0.0035**	**0.02205**	**0.043917**	**0.0512365**	**0.04303545**	**0.028759500**	**0.0161963065**
33			**0.0020**	**0.01745**	**0.039662**	**0.0504315**	**0.04521000**	**0.031861500**	**0.0187766131**
34			**0.0010**	**0.01340**	**0.035127**	**0.0488565**	**0.04682700**	**0.034835625**	**0.0214938745**
35			**0.0004**	**0.00996**	**0.030492**	**0.0465795**	**0.04782360**	**0.037594305**	**0.0243015388**

36	**0.0001**	**0.00715**	**0.025927**	**0.0436975**	**0.04816030**	**0.040051495**	**0.0271421810**
37		**0.00495**	**0.021582**	**0.0403305**	**0.04782360**	**0.042126975**	**0.0299515480**
38		**0.00330**	**0.017577**	**0.0366135**	**0.04682700**	**0.043750575**	**0.0326602870**
39		**0.00210**	**0.013992**	**0.0326865**	**0.04521000**	**0.044865975**	**0.0351966340**
40		**0.00126**	**0.010872**	**0.0286860**	**0.04303545**	**0.045433800**	**0.0374894389**
41		**0.00070**	**0.008232**	**0.0247380**	**0.04038560**	**0.045433800**	**0.0394713550**
42		**0.00035**	**0.006062**	**0.0209525**	**0.03735720**	**0.044865975**	**0.0410820025**
43		**0.00015**	**0.004332**	**0.0174195**	**0.03405600**	**0.043750575**	**0.0422709100**
44		**0.00005**	**0.002997**	**0.0142065**	**0.03059100**	**0.042126975**	**0.0430000450**
45		**0.00001**	**0.002002**	**0.0113575**	**0.02706880**	**0.040051495**	**0.0432457640**
46			**0.001287**	**0.0088935**	**0.02358840**	**0.037594305**	**0.0430000450**
47			**0.000792**	**0.0068145**	**0.02023680**	**0.034835625**	**0.0422709100**
48			**0.000462**	**0.0051030**	**0.01708575**	**0.031861500**	**0.0410820025**
49			**0.000252**	**0.0037290**	**0.01419000**	**0.028759500**	**0.0394713550**
50			**0.000126**	**0.0026544**	**0.01158684**	**0.025614639**	**0.0374894389**
51			**0.000056**	**0.0018368**	**0.00929672**	**0.022505751**	**0.0351966340**
52			**0.000021**	**0.0012327**	**0.00732474**	**0.019502505**	**0.0326602870**
53			**0.000006**	**0.0008001**	**0.00566280**	**0.016663185**	**0.0299515480**
54			**0.000001**	**0.0005005**	**0.00429220**	**0.011403331**	**0.0271421810**
55				**0.0003003**	**0.00318648**	**0.011645073**	**0.0243015388**
56				**0.0001716**	**0.00231429**	**0.009517662**	**0.0214938745**
57				**0.0000924**	**0.00164208**	**0.007658190**	**0.0187766131**
58				**0.0000462**	**0.00113640**	**0.006063255**	**0.0161963065**
59				**0.0000210**	**0.00076560**	**0.004720815**	**0.0137924380**
60				**0.0000084**	**0.00050100**	**0.003612231**	**0.0115921972**
61				**0.0000028**	**0.00031760**	**0.002714319**	**0.0096130540**
62				**0.0000007**	**0.00019440**	**0.002001285**	**0.0078629320**
63				**0.0000001**	**0.00011440**	**0.001446445**	**0.0063412580**
64					**0.00006435**	**0.001023660**	**0.0050402935**
65					**0.00003432**	**0.000708444**	**0.0039466306**
66					**0.00001716**	**0.000478731**	**0.0030427375**
67					**0.00000792**	**0.000315315**	**0.0023084500**
68					**0.00000330**	**0.000202005**	**0.0017223250**
69					**0.00000120**	**0.000125565**	**0.0012628000**
70					**0.00000036**	**0.000075501**	**0.0009091270**
71					**0.00000008**	**0.000043749**	**0.0006420700**
72					**0.00000001**	**0.000024310**	**0.0004443725**
73						**0.000012870**	**0.0003010150**
74						**0.000006435**	**0.0001992925**
75						**0.000003003**	**0.0001287484**
76						**0.000001287**	**0.0000810040**
77						**0.000000495**	**0.0000495220**
78						**0.000000165**	**0.0000293380**

79	**0.000000045**	**0.0000167860**
80	**0.000000009**	**0.0000092368**
81	**0.000000001**	**0.0000048620**
82		**0.0000024310**
83		**0.0000011440**
84		**0.0000005005**
85		**0.0000002002**
86		**0.0000000715**
87		**0.0000000220**
88		**0.0000000055**
89		**0.0000000010**
90		**0.0000000001**

APPENDIX “A2”

Table 8: Estimate of cumulative probability of sum of independent positive integers

	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
y	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**	**p(Y≤y)**
0	**0.01**	**0.001**	**0.0001**	**0.00001**	**0.000001**	**0.0000001**	**0.00000001**	**0.000000001**	**0.0000000001**
1	**0.03**	**0.004**	**0.0005**	**0.00006**	**0.000007**	**0.0000008**	**0.00000009**	**0.000000010**	**0.0000000011**
2	**0.06**	**0.010**	**0.0015**	**0.00021**	**0.000028**	**0.0000036**	**0.00000045**	**0.000000055**	**0.0000000066**
3	**0.10**	**0.020**	**0.0035**	**0.00056**	**0.000084**	**0.0000120**	**0.00000165**	**0.000000220**	**0.0000000286**
4	**0.15**	**0.035**	**0.0070**	**0.00126**	**0.000210**	**0.0000330**	**0.00000495**	**0.000000715**	**0.0000001001**
5	**0.21**	**0.056**	**0.0126**	**0.00252**	**0.000462**	**0.0000792**	**0.00001287**	**0.000002002**	**0.0000003003**
6	**0.28**	**0.084**	**0.0210**	**0.00462**	**0.000924**	**0.0001716**	**0.00003003**	**0.000005005**	**0.0000008008**
7	**0.36**	**0.120**	**0.0330**	**0.00792**	**0.001716**	**0.0003432**	**0.00006435**	**0.000011440**	**0.0000019448**
8	**0.45**	**0.165**	**0.0495**	**0.01287**	**0.003003**	**0.0006435**	**0.00012870**	**0.000024310**	**0.0000043758**
9	**0.55**	**0.220**	**0.0715**	**0.02002**	**0.005005**	**0.0011440**	**0.00024310**	**0.000048620**	**0.0000092378**
10	**0.64**	**0.283**	**0.0997**	**0.02998**	**0.008002**	**0.0019441**	**0.00043750**	**0.000092369**	**0.0000184746**
11	**0.72**	**0.352**	**0.1345**	**0.04338**	**0.012334**	**0.0031768**	**0.00075510**	**0.000167870**	**0.0000352606**
12	**0.79**	**0.425**	**0.1760**	**0.06083**	**0.018396**	**0.0050136**	**0.00125610**	**0.000293435**	**0.0000645986**
13	**0.85**	**0.500**	**0.2240**	**0.08288**	**0.026628**	**0.0076680**	**0.00202170**	**0.000495440**	**0.0001141206**
14	**0.90**	**0.575**	**0.2780**	**0.10998**	**0.037500**	**0.0113970**	**0.00315810**	**0.000810755**	**0.0001951246**
15	**0.94**	**0.648**	**0.3372**	**0.14244**	**0.051492**	**0.0165000**	**0.00480018**	**0.001289486**	**0.0003238730**
16	**0.97**	**0.717**	**0.4005**	**0.18039**	**0.069069**	**0.0233145**	**0.00711447**	**0.001997930**	**0.0005231655**
17	**0.99**	**0.780**	**0.4665**	**0.22374**	**0.090651**	**0.0322080**	**0.01030095**	**0.003021590**	**0.0008241805**
18	**1.00**	**0.835**	**0.5335**	**0.27214**	**0.116578**	**0.0435655**	**0.01459315**	**0.004468035**	**0.0012685530**
19		**0.880**	**0.5995**	**0.32494**	**0.147070**	**0.0577720**	**0.02025595**	**0.006469320**	**0.0019106230**
20		**0.916**	**0.6628**	**0.38125**	**0.018220**	**0.0751915**	**0.02758069**	**0.009183639**	**0.0028197500**
21		**0.944**	**0.7220**	**0.44000**	**0.221859**	**0.0961440**	**0.03687741**	**0.012795870**	**0.0040825500**
22		**0.965**	**0.7760**	**0.50000**	**0.265776**	**0.1208820**	**0.04846425**	**0.017516685**	**0.0058048750**
23		**0.980**	**0.8240**	**0.56000**	**0.313488**	**0.1495680**	**0.06265425**	**0.023579940**	**0.0081133250**
24		**0.990**	**0.8655**	**0.61875**	**0.364365**	**0.1822545**	**0.07974000**	**0.031238130**	**0.0111560625**
25		**0.996**	**0.9003**	**0.67506**	**0.417627**	**0.2188680**	**0.09997680**	**0.040755792**	**0.0151026931**
26		**0.999**	**0.9285**	**0.72786**	**0.472374**	**0.2591985**	**0.12356520**	**0.052400865**	**0.0201429866**
27		**1.000**	**0.9505**	**0.77626**	**0.527626**	**0.3028960**	**0.15063400**	**0.066434170**	**0.0264842446**
28			**0.9670**	**0.81961**	**0.582373**	**0.3494755**	**0.18122500**	**0.083097355**	**0.0343471766**
29			**0.9790**	**0.85756**	**0.635635**	**0.3983320**	**0.21528100**	**0.102599860**	**0.0439602306**
30			**0.9874**	**0.89002**	**0.686512**	**0.4487635**	**0.25263820**	**0.125105611**	**0.0555524278**
31			**0.9930**	**0.91712**	**0.734224**	**0.5000000**	**0.29302380**	**0.150720250**	**0.0693448658**
32			**0.9965**	**0.93917**	**0.778141**	**0.5512365**	**0.33605925**	**0.179479750**	**0.0855411723**
33			**0.9985**	**0.95662**	**0.817803**	**0.6016680**	**0.38126925**	**0.211341250**	**0.1043173033**
34			**0.9995**	**0.97002**	**0.852930**	**0.6505245**	**0.42809625**	**0.246176875**	**0.1258111778**
35			**0.9999**	**0.97998**	**0.883422**	**0.6971040**	**0.47591985**	**0.283771180**	**0.1501127166**

36	**1.0000**	**0.98713**	**0.909349**	**0.7408015**	**0.52408015**	**0.323822675**	**0.1772548976**
37		**0.99207**	**0.930931**	**0.7811320**	**0.57190375**	**0.365949650**	**0.2072064456**
38		**0.99538**	**0.948508**	**0.8177455**	**0.61873075**	**0.409700225**	**0.2398667326**
39		**0.99748**	**0.962500**	**0.8504320**	**0.66394075**	**0.454566200**	**0.2750633666**
40		**0.99874**	**0.973372**	**0.8791180**	**0.70697620**	**0.500000000**	**0.3125528055**
41		**0.99944**	**0.981604**	**0.9038560**	**0.74736180**	**0.545433800**	**0.3520241605**
42		**0.99979**	**0.987666**	**0.9248085**	**0.78471900**	**0.590299775**	**0.3931061630**
43		**0.99994**	**0.991998**	**0.9422280**	**0.81877500**	**0.634050350**	**0.4353770730**
44		**0.99999**	**0.994995**	**0.9564345**	**0.84936600**	**0.676177325**	**0.4783771180**
45		**1.00000**	**0.996997**	**0.9677920**	**0.87643480**	**0.716228820**	**0.5216228820**
46			**0.998284**	**0.9766855**	**0.90002320**	**0.753823125**	**0.5646229270**
47			**0.999076**	**0.9835000**	**0.92026000**	**0.788658750**	**0.6068938370**
48			**0.999538**	**0.9886030**	**0.93734575**	**0.820520250**	**0.6479758395**
49			**0.999790**	**0.9923320**	**0.95153575**	**0.849279750**	**0.6874471945**
50			**0.999916**	**0.9949864**	**0.96312259**	**0.874894389**	**0.7249366334**
51			**0.999972**	**0.9968232**	**0.97241931**	**0.897400140**	**0.7601332674**
52			**0.999993**	**0.9980559**	**0.97974405**	**0.916902645**	**0.7927935544**
53			**0.999999**	**0.9988560**	**0.98540685**	**0.933565830**	**0.8227451024**
54			**1.000000**	**0.9993565**	**0.98969905**	**0.947599135**	**0.8498872834**
55				**0.9996568**	**0.99288553**	**0.959244208**	**0.8741888222**
56				**0.9998284**	**0.99519982**	**0.968761870**	**0.8956826967**
57				**0.9999208**	**0.99684190**	**0.976420060**	**0.9144588277**
58				**0.9999670**	**0.99797830**	**0.982483315**	**0.9306551342**
59				**0.9999880**	**0.99874390**	**0.987204130**	**0.9444475722**
60				**0.9999960**	**0.99924490**	**0.990816361**	**0.9560397694**
61				**0.9999992**	**0.99956250**	**0.993530680**	**0.9656528234**
62				**0.9999999**	**0.99975690**	**0.995531965**	**0.9735157554**
63				**1.0000000**	**0.99987130**	**0.996978410**	**0.9798570134**
64					**0.99993565**	**0.998002070**	**0.9848973069**
65					**0.99996997**	**0.998710514**	**0.9888439375**
66					**0.99998713**	**0.999189245**	**0.9918866750**
67					**0.99999505**	**0.999504560**	**0.9941951250**
68					**0.99999835**	**0.999706565**	**0.9959174500**
69					**0.99999955**	**0.999832130**	**0.9971802500**
70					**0.99999991**	**0.999907631**	**0.9980893770**
71					**0.99999999**	**0.999951380**	**0.9987314470**
72					**1.00000000**	**0.999975690**	**0.9991758195**
73						**0.999988560**	**0.9994768345**
74						**0.999994995**	**0.9996761270**
75						**0.999997998**	**0.9998048754**
76						**0.999999285**	**0.9998858794**
77						**0.999999780**	**0.9999354014**
78						**0.999999945**	**0.9999647394**

79	**0.999999990**	**0.9999815254**
80	**0.999999999**	**0.9999907622**
81	**1.000000000**	**0.9999956242**
82		**0.9999980552**
83		**0.9999991992**
84		**0.9999996997**
85		**0.9999998999**
86		**0.9999999714**
87		**0.9999999934**
88		**0.9999999989**
89		**0.9999999999**
90		**1.0000000000**