Ilorin Journal of Science
Volume 4, Number 1, 2017, pp. 42– 46 (Printed in Nigeria)
ISSN: 2408 – 4840 © 2017 Faculty of Physical Sciences, University of Ilorin
https://doi.org/10.54908/iljs.2017.04.01.003

ILORIN
JOURNAL OF SCEINCE

ILJS-17-006

Coefficient Bounds for the Functions in the Class of Sigmoid Function

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Abstract

In this work, coefficient bounds for the functions in the class $S(\lambda, \phi)$ were obtained. This work was concluded by determining Fekete-Szego functional and the Hankel determinant.

Keywords: Sigmoid function, Subordination, Fekete-Szego Inequality, Analytic Functions,Univalent Functions.

1. Introduction

Let U be a unit disc in $\{z \in C: |z| < 1\}$. Let S be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \; (z \in \mathbf{C}), \tag{1}$$

which are analytic and univalent in U and satisfying the conditions $f(0) = 0$ and $f'(0) = 1$.
Let the functions f and g be analytic in U. Then f is said to be subordinate to g, written as $f \prec g$, if there exists a function ω analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$, and such that $f(z) = g(\omega(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (Pommerenke,1975).

The sigmoid function

$$h(z) = \frac{1}{1 + e^{-z}}$$

is differentiable and has the following properties:

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- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one – to – one function.
- It increases monotonically.

Definition 1: The set ***P*** is the set of all functions of the form:
$P(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$, that are analytic in ***U*,** and such that for $z \epsilon \boldsymbol{U}$, $Re(P(z)) > 0$ (Duren, 1983; Goodman, 1983).

Definition 2: Let *h(z)* be a sigmoid function and

$$\phi(z) = 2h(z) = \frac{2}{1+e^{-z}} = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \cdots \tag{2}$$

where $\phi(z)$ is the modified sigmoid function which belongs to class ***P*** (Fadipe-Joseph *et al.*, 2013).

Altinkaya and Yalcin (2016) considered a subclass of univalent functions and obtained coefficients expansion using Chebyshev polynomials. Ramachandran and Dhanalakshmi (2017a) established coefficient estimates for a class of spiral-like functions in the space of sigmoid function. Ramachandran and Dhanalakshmi (2017b) obtained the Fekete- Szego functional for a subclass of analytic functions related to sigmoid function. Here, coefficient bounds for a class of univalent function using sigmoid polynomials were obtained.

2. Main Results

Here, the main results are given.

Definition 3: A function $f \epsilon \boldsymbol{A}$ is said to be in the class $\boldsymbol{S(\lambda, \phi)}$, $0 \leq \lambda \leq 1$, if the following subordination holds:

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z) \tag{3}$$

The following lemma will be required for the proof of the main results.

Lemma 1: If $\omega \epsilon \Omega$, $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$, ($z \epsilon \boldsymbol{U}$), then

$$|c_n| \leq 1 \qquad n = 1, 2, \ldots, \qquad |c_2| \leq 1 - |c_1|^2 \tag{4}$$

and

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}. \tag{5}$$

The result is sharp, the functions $\omega(z) = z$, $\omega_a(z) = z\frac{z+a}{1+\bar{a}z}$ (z ϵ ***U***, |a| < 1) are extremal functions (Keogh and Merkes, 1969).

Theorem 1: If *f(z)* belongs to the class ***S(λ, ϕ)***, then

$$|a_2| \leq \frac{1}{2(1+\lambda)},$$

$$|a_3| \leq \frac{1}{4(1+2\lambda)} + \frac{1+3\lambda}{8(1+\lambda)^2(1+2\lambda)},$$

$$|a_4| \leq \frac{13}{72(1+3\lambda)} + \frac{1+7\lambda}{24(1+\lambda)^3(1+2\lambda)} + \frac{1+5\lambda}{8(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{1+5\lambda}{2(1+\lambda)^3(1+2\lambda)},$$

$$|a_5| \leq \frac{5}{32(1+4\lambda)} + \frac{1+7\lambda}{16(1+\lambda)(1+4\lambda)}\left\{\frac{1}{9(1+3\lambda)} + \frac{1+7\lambda}{3(1+\lambda)^3(1+3\lambda)} + \frac{1+5\lambda}{(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{1+5\lambda}{2(1+\lambda)^3(1+2\lambda)}\right\} + \frac{1+11\lambda}{16(1+\lambda)^2(1+2\lambda)(1+4\lambda)}\left\{1 + \frac{1+3\lambda}{2(1+\lambda)}\right\} + \frac{1+8\lambda}{8(1+\lambda)^2(1+4\lambda)}\left\{\frac{1}{4} + \frac{1+3\lambda}{(1+2\lambda)^2} + \frac{(1+3\lambda)^2}{16(1+\lambda)^2(1+2\lambda)^2}\right\} + \frac{1+15\lambda}{64(1+\lambda)^4(1+4\lambda)}.$$

Proof: Let *f* ϵ ***S(λ, ϕ).*** From (3), we have

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{1}{2}\omega(z) - \frac{1}{24}\omega^3(z) + \frac{1}{240}\omega^5(z) - \cdots \tag{6}$$

for some analytic function $\omega(z)$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all *z* ϵ ***U.***

Therefore,

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{c_1}{2}z + \frac{c_2}{2}z^2 + \left(\frac{c_3}{2} - \frac{c_1^3}{24}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1^2c_2}{8}\right)z^4 + \cdots \tag{7}$$

And, it follows that:

$$(1+\lambda)a_2 = \frac{c_1}{2}, \tag{8}$$

$$2(1+2\lambda)a_3 - (1+3\lambda)a_2^2 = \frac{c_2}{2}, \tag{9}$$

$$3(1+3\lambda)a_4 - 3(1+5\lambda)a_2a_3 + (1+7\lambda)a_2^3 = \frac{c_3}{2} - \frac{c_1^3}{24}, \tag{10}$$

$$4(1+4\lambda)a_5 - 4(1+7\lambda)a_2a_4 + 4(1+11\lambda)a_2^2a_3 - 2(1+8\lambda)a_2^3 - (1+15\lambda)a_2^4 = \frac{c_4}{2} - \frac{c_1^2c_2}{8}. \tag{11}$$

Then, from (4), the result follows.

Corollary 1: If *f(z)* belongs to the class ***S(0, ϕ)***, then

$$|a_2| \leq \frac{1}{2},\ |a_3| \leq \frac{3}{8},\ |a_4| \leq \frac{59}{144},\ |a_5| \leq \frac{635}{1152}.$$

Corollary 2: If *f(z)* belongs to the class ***S(1, ϕ)***, then

$$|a_2| \leq \frac{1}{4},\ |a_3| \leq \frac{1}{8},\ |a_4| \leq \frac{59}{576},\ |a_5| \leq \frac{179}{1440}.$$

Theorem 2 (Fekete – Szegö Inequality): If *f(z)* belongs to the class ***S(λ, ϕ)***, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{4(1+\lambda)}\left(\frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)}\right) & \mu = 0 \\ \frac{1}{4(1+\lambda)}\left(\frac{1}{(1+\lambda)} - \frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)}\right) & \mu = 1 \\ \frac{1}{4(1+\lambda)}\left(\frac{\mu}{(1+\lambda)} - \frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)}\right) & 0 < \mu < 1 \end{cases}.$$

Furthermore

$$|a_2a_4 - a_3^2| \le \frac{13}{144(1+\lambda)(1+3\lambda)} + \frac{1+5\lambda}{16(1+\lambda)^2(1+2\lambda)(1+3\lambda)} + \frac{1+5\lambda}{32(1+\lambda)^4(1+2\lambda)} + \frac{1+7\lambda}{48(1+\lambda)^4(1+3\lambda)} +$$
$$\frac{1}{16(1+2\lambda)^2} + \frac{1+3\lambda}{4(1+\lambda)^2(1+2\lambda)^2} + \frac{(1+3\lambda)^2}{64(1+\lambda)^4(1+2\lambda)^2}.$$

Proof: From (7) and (8), we have

$$|a_3 - \mu a_2^2| = \left|\frac{c_2}{4(1+\lambda)} + \frac{(1+3\lambda)c_1^2}{8(1+\lambda)^2(1+2\lambda)} - \frac{\mu c_1^2}{4(1+\lambda)^2}\right|$$
$$= \left|\frac{1}{4(1+\lambda)}\left(c_2 - c_1^2\left(\frac{\mu}{(1+\lambda)} - \frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)}\right)\right)\right|.$$

Then, from (4), we have

$$|a_3 - \mu a_2^2| \le \frac{1}{4(1+\lambda)} max\left\{1, \left|\frac{\mu}{(1+\lambda)} - \frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)}\right|\right\}.$$

Now, for the functional $|a_2a_4 - a_3^2|$, substituting for a_2, a_3, a_4, we have

$$|a_2a_4 - a_3^2| \le \frac{13}{144(1+\lambda)(1+3\lambda)} + \frac{1+5\lambda}{16(1+\lambda)^2(1+2\lambda)(1+3\lambda)} + \frac{1+5\lambda}{32(1+\lambda)^4(1+2\lambda)} + \frac{1+7\lambda}{48(1+\lambda)^4(1+3\lambda)} +$$
$$\frac{1}{16(1+2\lambda)^2} + \frac{1+3\lambda}{4(1+\lambda)^2(1+2\lambda)^2} + \frac{(1+3\lambda)^2}{64(1+\lambda)^4(1+2\lambda)^2},$$

which is the desired result.

Acknowledgements

The authors acknowledge the comments of the reviewers for their suggestions and comments in improving the quality of the manuscripts. The first author acknowledges the Abdus Salam International Center for Theoretical Physics (ICTP), Italy for the Associateship award.

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