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Construction of Orthogonal Polynomial as Basis Function for Solving Fractional Order Integro Differential Equations

Uwaheren[1]*, O. A and Taiwo[1], O. A.
[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria

Abstract
This article is concerned with the use of orthogonal polynomials as basis functions to solve fractional order Integro-differential equations. Standard collocation method was used. We constructed orthogonal polynomials by assuming a quadratic weight function that is positive in the interval of consideration; [0, 1]. The assumed approximate function was substituted into the general class of fractional order Integro-differential equations and after some simplifications; the resulted equation was then collocated at equally spaced interior points of the interval. The resulted system of algebraic equations is solved to obtain the unknown constants. The constants are substituted back into the assumed approximate solution to get the required approximate solution.

Keyword: Orthogonal functions, fractional order, Standard collocation method and fractional Integro-differential equations.

1. Introduction

Most formulations of mathematical models of physical problems lead to linear or nonlinear differential equations. Many of these differentials equations do not have analytical solution or that their solutions are difficult to find in a closed form. To solve these types of equations, we have to employ numerical techniques and numerical standard collocation methods have been found to be very useful. The method usually involves evaluating equation at equally spaced points to achieve a system of algebraic equations. Several numerical methods to solve fractional differential equations and fractional integro differential equations have been given by a number of researchers. Atabakzadeh *et al*. (2012), Hashim *et al*. (2009), Saadatmandi and Dehghan (2010) and Ibtisan (2011) applied Chebychev operational matrix method and Homotopy analysis methods to solve various forms of fractional differential equations and fractional integro differential equations. Atabakzadeh *et al*. (2012) particularly used Chebyshev operational matrix method to solve multi order fractional ordinary differential equations. By using shifted Chebyshev polynomial, they were able to obtain a satisfactory

*Corresponding author: Uwaheren, O. A.
Email: uwaheren.ao@unilorin.edu.ng

result without linearization. Fukang et al., 2014 used a modified Variational Iteration Method and fractional-order Legendre functions method to solve non-linear fractional-order differential equations.

The paper used Legendre basis functions as operational matrices of orthogonal functions to integrate and differentiate fractional differential equations. Mohammed (2014) used Least Squares method and shifted Chebyshev polynomial to solve linear fractional integro differential equations. Omar (2014) considered fractional integro differential equation as very important integro differential equations where the differentiation and the integration appearing in the equations are of non-integer order. Omar solved some multi fractional order Integro-differential equations using Variation Iterative Method (VIM). Taiwo and Odetunde (2013) used Iterative decomposition method to solve multi-term fractional differential equation.

Many other researchers including Taiwo and Uwaheren (2015) employed Collocation method to solve different problems of fractional integro-differential equations using Power series, Legendre, Chebyshev or Hermite polynomials as basis functions. Yousefi *et al*. (2017) constructed operational matrix of fractional integration of hybrid functions and combined the features of the hybrid functions and their operational matrix to reduce fractional differential and integro-differential equations to system of algebraic equations to obtain the required solution. According to Gradimir (1991), orthogonal functions have received considerable attention in dealing with various problems of fractional differential equations. In this work, orthogonal polynomial functions are constructed using a quadratic weight function, $w(t)= t^2 + t -1$ and applied to solve fractional order Integro-differential equations.

1.1 Basic relevant definitions to the work

Here, we present some basic relevant definitions which are very useful in this work.

Definition 1

Fractional derivative in the Caputo sense is given as

$$\left(D_*^\alpha f\right)(x) = \left(J_\alpha^{n-\alpha}\right) D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_\alpha^x (x-t)^{n-\alpha-1} \left(\frac{d}{dx}\right)^n f(t)dt \qquad (1)$$

The operator D_a^α is defined by

$$\left(D_a^{\alpha} f\right)(x) = D^n \left(J_a^{n-\alpha}\right) f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_{\alpha}^{x} (x-t)^{n-\alpha-1} f(t) dt \qquad (2)$$

is called Riemann-Liouvilles differential operator of order α for $a \leq x \leq b$.

Definition 2:

Fractional order Integro-differential equation:- The general form of fractional order integro- differential equation considered in this paper is given as:

$$D_*^{\alpha} y(x) = f(x) + \int_0^1 K(x,t) y(t) dt \qquad 0 < x, t < 1 \qquad (3)$$

together with the following supplementary condition, $y(0) = \eta$ and D^{α} is in the Caputo sense of the differential integral functions and α is a parameter denoting the fractional order derivative of the function.

A very important property of $D^{\alpha} f(t)$ is:

$$D_*^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha} \qquad (4)$$

Definition 3:

Orthogonal functions: Two different functions say y_n (x) and y_m (x) are said to be orthogonal if their inner product is zero when n is not equal to m.

$$\langle y_n(x), y_m(x) \rangle \equiv \int_a^b y_n(x) y_m(x) dx = 0 \qquad (5)$$

On the other hand, a third function $w(x) > 0$ exists, then:

$$\langle y_n(x), y_m(x) \rangle = \int_a^b w(x) y_n(x) y_m(x) dx = 0 \qquad (6)$$

Then we say that y_n (x) and y_m (x) are mutually orthogonal with respect to the weight function w(x). The construction of our polynomial actually followed the basic procedure for obtaining orthogonal polynomials but using a quadratic weight functions.

1.2 Construction of Orthogonal polynomials

Here, we are guided by the conditions of orthogonality of functions y_n (x) and y_m (x), to construct orthogonal polynomials using some basic quadratic weight functions. We state the conditions required as follows: Let the orthogonal polynomial be denoted Q_n (t), then

$$Q_n(t) = \sum_{i=0}^{n} c_i^{(n)} t^i$$

where $C_i^{(n)}$ is called the orthogonal polynomial coefficients.

The inner product $\langle Q_n(t) Q_m(t) \rangle = 0$, for $n \neq m$ and

$$Q_n(1) = Q_0(t) = 1 \tag{8}$$

for $n = 0$

$$Q_0(t) = \sum_{i=0}^{0} C_0^{(0)} t^i = 1 \tag{9}$$

for $n = 1$

$$Q_1(t) = \sum_{i=0}^{1} C_0^{(1)} t^i = C_0^{(1)} + C_1^{(1)} t \tag{10}$$

$$\therefore Q_1(1) = 1, C_0^{(1)} + C_1^{(1)} = 1 \tag{11}$$

Taking the inner product of w(t) and Q_1 (t), we have

$$\langle w(t) Q_1(t) \rangle = 0 \Rightarrow \int_0^1 (t^2 - t + 1) \cdot (1) \cdot (C_0^{(1)} + C_1^{(1)} t) dt = 0 \tag{12}$$

Integrating (12) from 0 to 1, we have

$$\frac{5}{6} C_0^{(1)} + \frac{5}{12} C_1^{(1)} = 0 \tag{13}$$

Solving (11) and (13) we have that

$$C_0^{(1)} = -1 \text{ and } C_1^{(1)} = 2$$

$$\therefore Q_1(t) = -1 + 2x \text{ or } 2x - 1 \tag{15}$$

For n=2

$$Q_2(t) = \sum_{i=0}^{2} C_2^{(n)} t^i = C_0^{(2)} + C_1^{(2)} t + C_2^{(2)} t^2 \qquad (16)$$

$$Q_2(1) = C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \qquad (17)$$

Taking the inner product of w(t), Q_1 (t) and Q_2 (t)

$$\langle w(t)Q_1(t)Q_2(t)\rangle = 0 \Rightarrow \int_0^1 (t^2-t+1)\cdot(2t-1)\cdot(C_0^{(2)}+C_1^{(2)}t+C_2^{(2)}t^2)dt = 0 \quad (18)$$

Integrating (18) from 0 to 1, we have

$$0C_0^{(2)} + \frac{3}{20}C_1^{(2)} + \frac{3}{20}C_2^{(2)} = 0 \qquad (19)$$

Taking the inner product w(t), Q_0 (t) and Q_2 (t)

$$\langle w(t)Q_0(t)Q_2(t)\rangle = 0 \ \Rightarrow \int_0^1 (t^2-t+1)\cdot(1)\cdot(C_0^{(2)}+C_1^{(2)}t+C_2^{(2)}t^2)dt = 0 \quad (20)$$

Integrating (20) from 0 to 1, we have

$$\frac{5}{6}C_0^{(2)} + \frac{5}{12}C_1^{(2)} + \frac{17}{60}C_2^{(2)} = 0 \qquad (21)$$

Solving (17), (19) and (21) we have

$$C_0^{(2)} = 1 C_1^{(2)} = \frac{-25}{4} \text{ and } C_2^{(2)} = \frac{25}{4}$$

therefore,

$$Q_2(t) = \frac{1}{4}(25t^2 - 25t + 4) \qquad (22)$$

For $n = 3$

$$Q_3(t) = \sum_{i=0}^{3} C_3^{(n)} t^i = C_0^{(3)} + C_1^{(3)} t + C_2^{(3)} t^2 + C_3^{(3)} t^3 \qquad (23)$$

$$Q_3(1) = 1 \Rightarrow C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \qquad (24)$$

Taking the inner products of w(t), Q_0 (t) and Q_3 (t)

$$\langle w(t)Q_0(t)Q_3(t)\rangle = 0 \;\Rightarrow\; \int_0^1 (t^2 - t + 1)\cdot(1)\cdot(C_0^{(3)} + C_1^{(3)}t + C_2^{(3)}t^2 + C_3^{(3)}t^3)dt = 0 \quad (25)$$

Integrating (25) from 0 to 1, we have

$$\frac{5}{6}C_0^{(3)} + \frac{5}{12}C_1^{(3)} + \frac{17}{60}C_2^{(3)} + \frac{13}{60}C_3^{(3)} = 0 \quad (26)$$

$$\langle w(t)Q_1(t)Q_3(t)\rangle = 0 \;\Rightarrow\; \int_0^1 (t^2 - t + 1)\cdot(2t-1)\cdot(C_0^{(3)} + C_1^{(3)}t + C_2^{(3)}t^2 + C_3^{(3)}t^3)dt = 0 \quad (27)$$

Integrating (23) from 0 to 1, we have

$$\frac{3}{20}C_1^{(3)} + \frac{3}{20}C_2^{(3)} + \frac{19}{140}C_3^{(3)} = 0 \quad (28)$$

$$\langle w(t)Q_2(t)Q_1(t)\rangle = 0 \Rightarrow \int_0^1 (t^2 - t + 1).(\frac{25t^2 - 25t + 4}{4}).(c_0^{(3)} + c_1^{(3)}t + c_2^{(3)}t^2 + c_3^{(3)}t^3)dt = 0 \ldots(29)$$

Integrating (29) from 0 to 1, we have

$$\frac{17}{560}C_2^{(3)} + \frac{51}{11120}C_3^{(3)} = 0 \quad (30)$$

Solving (24), (26), (28) and (30), we have

$C_0^{(3)} = -1$, $C_1^{(3)} = 25/2$, $C_0^{(3)} = -63/2$ and $C_3^{(3)} = 21$.

Therefore

$Q_3(t) = \frac{1}{2}(42t^3 - 63t^2 + 25t - 2)$ and the process continues for

$$Q_n(x), n = 0, 1, 2, \ldots.$$

The first few terms of the orthogonal polynomial constructed here therefore using the weight function $w(t) = x^2 - x + 1$, $0 \le x \le 1$ are given as

$$
\begin{aligned}
Q_0(x) &= 1\\
Q_1(x) &= (2x-1)\\
Q_2(x) &= \frac{1}{4}(25x^2-25x+4)\\
Q_3(x) &= \frac{1}{2}(42x^3-63x^2+25x-2)\\
Q_4(x) &= \frac{1}{29}(2142x^4-4284x^3+2744x^2-602x+29)
\end{aligned}
\tag{32}
$$

Equation (32) satisfies the Orthogonal conditions:

$Q_n(0) = (-1)^n, Q_n(1) = 1$ and

$$\int_0^1 Q_n(x)Q_m(x)dx = \begin{cases} 0, & n \neq m \\ \frac{1}{2n+1}, & n = m \end{cases} \tag{33}$$

2. Methodology

The method used here assumed an approximate solution in terms of the constructed orthogonal polynomials as the basis functions. The assumed solution is substituted into the general class of fractional order integro differential equations and after simplification; the resulting algebraic linear equation is then collocated at equally spaced interior points on the interval of consideration. Thus the resulting algebraic linear system of equations is then solved by maple 18 or by Gaussian elimination method to obtain the unknown constants. The constants obtained are substituted into the assumed approximate solution to get the approximate solution. Without lost of generality, we assume an approximate solution of the form:

$$y_N(x) = \sum_{i=0}^{N} c_i Q_i(x) \tag{34}$$

where c_i, i =0 (1) n are unknown constants to be determined and $Q_i(t)$ are the orthogonal polynomials constructed using the weight function, w(t) earlier mentioned. Substituting (34) into (32), we have

$$D_*^{\alpha}\sum_{i=0}^{N}(c_iQ_i(x)) = f(x) + \int_0^1 (\sum_{i=0}^{N}(c_iQ_i(t)))dt \tag{35}$$

Thus equation (35) is then simplified further to obtain:

$$[D_*^\alpha(Q_0(x)) - \int_0^1 (D_*^\alpha(Q_0(t))dt]c_0 + [D_*^\alpha(Q_1(x)) - \int_0^1 (D_*^\alpha(Q_1(t))dt]c_1$$
$$+[D_*^\alpha(Q_2(x)) - \int_0^1 (D_*^\alpha(Q_2(t))dt]c_2 + [D_*^\alpha(Q_3(x)) - \int_0^1 (D_*^\alpha(Q_3(t))dt]c_3 + \cdots +$$
$$[D_*^\alpha(Q_{N-1}(x)) - \int_0^1 (D_*^\alpha(Q_{N-1}(t))dt]c_{N-1} + [D_*^\alpha(Q_N(x)) - \int_0^1 (D_*^\alpha(Q_0(t))dt]c_N = f(x) \quad (36)$$

where, $Q_i(x)$, i =0 (1) N are the orthogonal polynomials constructed above.

Equation (36) is then collocated at equally spaced interior interval;

$t_i = a + \frac{(b-a)i}{N}$, (i $= 0(1)$N), to give (N+1) linear equations in (N+1) unknown constants which are put in matrix form:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2m} \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ A_{m1+1} & A_{m2+1} & A_{m3+1} & \cdots & A_{mm+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_{m+1} \end{pmatrix} = \begin{pmatrix} B_{11} \\ B_{22} \\ B_{33} \\ \vdots \\ \vdots \\ Bm+1 \end{pmatrix} \quad (37)$$

where

$A_{11} = D_*^\alpha(Q_0(x)) - \int_0^1 (D_*^\alpha(Q_0(t)))dt$, $A_{12} = D_*^\alpha(Q_1(x)) - \int_0^1 (D_*^\alpha(Q_1(t)))dt$

etc.

The (N+1) linear equations are then solved using the Gaussian elimination method or any suitable computer package like maple 18 to obtain the unknown constants $c_i, (i = 0(1)N)$, and these values are then substituted back into the approximate solution to give the required approximate solution.

3. Numerical Examples

In this section some numerical examples linear fractional integro differential equations are presented to illustrate the method.

Example 1:

Consider the following fractional integro differential equation:

$$D^{0.5}y(x) = \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtQ(t)dt \qquad 0 < x, t < 1 \qquad (38)$$

Exact solution is
$y(x) = x^2 - x$

Equation (38) re-written as

$$D^{0.5}y(x) - \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} - \frac{x}{12} - \int_0^1 xtQ(t)dt = 0 \quad (39)$$

Applying (4) and (34) in (38), with some simplifications, we have

$$-\frac{8}{3}x^2+2x-.1142249373c_4x^{\frac{3}{2}}-.1683833127c_3x^{\frac{3}{2}}-.2707918771c_2x^{\frac{3}{2}}-.8862279615x^{\frac{3}{2}}c_0 -.4923488675x^{\frac{3}{2}}c_1-.1477046602x^{\frac{3}{2}}+c_0+\frac{8}{3}c_1x-\frac{1}{3}c_1+\frac{20}{3}x^2c_2-\frac{10}{3}c_2x+\frac{1}{6}c_2+\frac{448}{25}x^3c_3 -\frac{84}{5}x^2c_3+\frac{18}{5}c_3x-\frac{1}{10}c_3+\frac{256}{5}x^4c_4-\frac{1792}{25}x^3c_4+\frac{448}{15}x^2c_4-\frac{56}{15}c_4x+\frac{1}{15}c_4=0 \quad (40)$$

Equation (40) is collocated at equally spaced points of [0, 1], to give 5 linear algebraic equations which are then solved to give the unknown constants as:

$$c_0 = -.1500000010, c_1 = -.2499999984, c_2 = .3999999979, c_3 = 1.89005559610^{-9}, c_4 = -8.14739690210^{-10}$$

These values are then substituted into the assumed approximate solution. After further simplifications, we obtained the required approximate solution.

$$y_4(x) = -2.12332153910^{-9}-.9999999895x+.9999999737x^2+2.88344804010^{-8}x^3 -1.14063556610^{-8}x^4$$

Example 2:

Consider the following fractional integro differential equation:

$$D^{\frac{5}{3}}Q(x) = \frac{3\sqrt{3}\Gamma(\frac{3}{2})x^{\frac{1}{3}}}{\pi} - \frac{x^2}{5} - \frac{x}{4} + \int_0^1 (xt + x^2t^2)Q(t)dt \qquad 0 < x, t < 1 \quad (41)$$

Exact solution is $y(x) = x^2$

We rewrite (41) as

$$D^{\frac{5}{3}}Q(x) - \int_0^1 (xt + x^2t^2)Q(t)dt = \frac{3\sqrt{3}\Gamma(\frac{3}{2})x^{\frac{1}{3}}}{\pi} - \frac{x^2}{5} - \frac{x}{4} \quad (42)$$

Applying (4) and (34) in (42), we have

$$-.7817773893c_0+1.563554779c_1x+.2605924631c_1+17.58999126c_2x^2-1.954443473c_2x$$
$$-.1302962316c_2+88.65355596x^3c_3-44.32677798c_3x^2+2.110798951c_3x+0.7817773893e-1c_3$$
$$+379.9438112x^4c_4-354.6142238x^3c_4+78.80316085c_4x^2-2.188976690c_4x-0.5211849263e-1c_4$$
$$=7.035996504x^2+.1955022222x^{\frac{11}{3}}c_4+.2024844444x^{\frac{8}{3}}c_4+.2827800000x^{\frac{11}{3}}c_3+.2984900000x^{\frac{8}{3}}c_3$$
$$+.4800277778x^{\frac{8}{3}}c_2+.4363888889x^{\frac{11}{3}}c_2+1.571000000x^{\frac{8}{3}}c_0+1.047333333x^{\frac{11}{3}}c_0+.8727777778x^{\frac{8}{3}}c_1$$
$$+.6982222222x^{\frac{11}{3}}c_1-.6284000000x^{\frac{11}{3}}-.7855000000x^{\frac{8}{3}} \quad (43)$$

Equation (43) is collocated at equally spaced points of [0, 1], to give 5 linear algebraic equations which are then solved to give the unknown constants as

$$c_0=.1000000272, c_1=.5000000437, c_2=.3999999967, c_3=3.12809226210^{-9},$$
$$c_4=-1.58780014810^{-10}$$

These values are then substituted into the assumed approximate solution. After further simplifications, we obtained the required approximate solution.

$$1.17966054410^{-8}+6.9710^{-8}x+.9999999702x^2+2.10739890010^{-8}x^3-2.22292020710^{-9}x^4$$

Example 3:

Consider the following fractional integro differential equation:

$$D^{\frac{5}{6}}Q(x)=-\frac{3x^{\frac{1}{6}}}{91}\frac{\Gamma(\frac{5}{6})(-91+216x^2)}{\pi}+(5-2e)x+\int_0^1 (xe^t)Q(t)dt \qquad 0<x,t<1 \quad (44)$$

Exact solution is $y(x)=x-x^3$.

Following the same procedure, we have

$$D^{\frac{5}{6}}Q(x)-\int_0^1 (xe^t)Q(t)dt=-\frac{3x^{\frac{1}{6}}}{91}\frac{\Gamma(\frac{5}{6})(-91+216x^2)}{\pi}+(5-2e)x \qquad 0<x,t<1 \quad (45)$$

Applying (4) and (34) in (45), we have

$$\frac{.1795797629c_0}{x^{\frac{5}{6}}}+1.436638103c_1x^{\frac{1}{6}}-\frac{0.5985992098e-1c_1}{x^{\frac{5}{6}}}+4.617765332c_2x^{\frac{7}{6}}$$
$$-1.795797629c_2x^{\frac{1}{6}}+\frac{0.2992996048e-1c_2}{x^{\frac{5}{6}}}+14.32217678c_3x^{\frac{13}{6}}-11.63676864c_3x^{\frac{7}{6}}$$
$$+1.939461439c_3x^{\frac{1}{6}}-\frac{0.1795797629e-1c_3}{x^{\frac{5}{6}}}+45.22792669c_4x^{\frac{19}{6}}-57.28870715c_4x^{\frac{13}{6}}$$
$$+20.68758869c_4x^{\frac{7}{6}}-2.011293345c_4x^{\frac{1}{6}}+\frac{0.1197198419e-1c_4}{x^{\frac{5}{6}}}=-2.557531568x^{\frac{13}{6}}$$
$$+1.077478577x^{\frac{1}{6}}-.43656364x+1.71828182c_0x+.7605727270c_1x+.415418187c_2x$$
$$+.25823997c_3x+.1751767c_4x \quad (46)$$

Equation (46) is collocated at equally spaced points of [0, 1], to give 5 linear algebraic equations which are then solved to give the unknown constants as

$$c_0=.1999999843, c_1=.4285714238, c_2=-.4500000142, c_3=-.1785714229,$$
$$c_4=-2.10453036210^{-9}$$

These values are then substituted into the assumed approximate solution. After further simplifications, we obtained the required approximate solution.

$$-1.72203020210^{-8}+1.000000031x-9.610^{-8}x^2-.9999999211x^3-2.94634250710^{-8}x^4$$

Table 1: showing the results of the method on problem 1

x	Exact	Orthog Appx	Error
0.0	0.0000000000	-0.0000000021	2.1233e-09
0.1	-0.0900000000	-0.0900000013	1.3086e-09
0.2	-0.1600000000	-0.1600000008	8.6290e-10
0.3	-0.2100000000	-0.2100000006	6.5418e-10
0.4	-0.2400000000	-0.2400000006	5.7792e-10
0.5	-0.2500000000	-0.2500000006	5.5691e-10
0.6	-0.2400000000	-0.2400000006	5.4134e-10
0.7	-0.2100000000	-0.2100000004	5.0876e-10
0.8	-0.1600000000	-0.1600000004	4.6411e-10
0.9	-0.0900000000	-0.0900000005	4.3970e-10
1.0	0.0000000000	-0.0000000005	4.9520e-10

Table 2: showing the results of the method on problem 2

x	Exact	Orthog Appx	Error
0.0	0.0000000000	0.0000000118	1.1797e-08
0.1	0.0100000000	0.0100000185	1.8489e-08
0.2	0.0400000000	0.0400000247	2.4710e-08
0.3	0.0900000000	0.0900000306	3.0576e-08
0.4	0.1600000000	0.1600000361	3.6200e-08
0.5	0.2500000000	0.2500000417	4.1692e-08
0.6	0.3600000000	0.3600000472	4.7152e-08
0.7	0.4900000000	0.4900000527	5.2679e-08
0.8	0.6400000000	0.6400000584	5.8364e-08
0.9	0.8100000000	0.8100000643	6.4293e-08
1.0	1.0000000000	1.0000000710	7.0548e-08

Table 3: showing the results of the method on problem 3

x	Exact	Orthog Appx	Error
0.0	0.0000000000	-0.0000000172	1.7220e-08
0.1	0.0990000000	0.0989999850	1.5004e-08
0.2	0.1920000000	0.1919999858	1.4276e-08
0.3	0.2730000000	0.2729999854	1.4669e-08
0.4	0.3360000000	0.3359999840	1.5885e-08
0.5	0.3750000000	0.3749999824	1.7699e-08
0.6	0.3840000000	0.3839999800	1.9956e-08
0.7	0.3570000000	0.3569999775	2.2572e-08
0.8	0.2880000000	0.2879999745	2.5532e-08
0.9	0.1710000000	0.1709999711	2.8893e-08
1.0	0.0000000000	-0.0000000326	3.2784e-08

4. Conclusion

We applied our constructed orthogonal polynomials as basic functions to solve fractional order integro differential equations. Linear fractional order Integro-differential equations were solved and the results show a high level of convergence to the exact solution.

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