Ilorin Journal of Science
Volume 8, Number 1, 2021, pp. 1 – 20 (Printed in Nigeria)
ISSN: 2408 – 4840 © 2021 Faculty of Physical Sciences, University of Ilorin
https://doi.org/10.54908/iljs.2021.08.01.001

ILORIN
JOURNAL OF SCEINCE

ILJS-21-007

Free Vibration Analysis of Non-uniform Rayleigh Beams on Variable Winkler Elastic Foundation using Differential Transform Method

Olotu[1*], O. T., Agboola[2], O. O. and Gbadeyan[1], J. A.

[1] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2] Department of Mathematics, Covenant University, Ota, Nigeria.

Abstract

This study examines the effect of variable Winkler foundation on the natural frequencies of a prestressed non-uniform Rayleigh beam. In this work, the elastic coefficients of the foundations are assumed to vary along the length direction of the beam. A semi-analytical approach known as Differential Transform Method (DTM) is applied to the non-dimensional form of the governing equations of motion of the prestressed non-uniform Rayleigh beam and a set of recursive algebraic equations are obtained. Evaluating these derived equations and using some computer codes written and implemented in MAPLE 18, the non-dimensional frequencies and the associated mode shapes of the beam are obtained. The effects of variable Winkler foundation variations and axial force for various values of the slenderness ratio on the non-dimensional frequencies are investigated. The clamped-clamped and simply supported boundary conditions are considered to illustrate the accuracy and efficiency of this method. Finally, the results obtained are validated and are found to compare favorably well with those in the open literature.

Keyword: Free vibration, natural frequency, Winkler foundation variations and differential transform method.

1. Introduction

The problem of analyzing the vibration behaviour of beams resting on elastic foundations has a wide application in the analysis and design of the foundations of buildings, highways, railways and a host of other geotechnical structures. In fact, it is an important aspect of structural and geotechnical investigation. As a matter of fact, different beam theories namely Euler-Bernoulli, Timoshenko, Shear and Rayleigh beam theories have been used by scholars in carrying out the mathematical formulation of beam vibration problems. Amongst the models of elastic foundations that have been used in the literature is the one-parameter model known as the Winkler model (Eisenberger and Clastornik, 1987; Ike, 2018).

Corresponding Author*: Olotu, O. T.
Email: olotu.ot@unilorin.edu.ng

Winkler foundation model, being the simplest of all the models, is the most widely used model because of its simplicity and for convenience's sake. The Winkler model assumes that the subgrade/foundation reaction is directly proportional to the beam deflection at any point on the foundation. In other words, the soil is modelled as uniformly distributed linear elastic vertical springs, which tend to produce distributed reactions along the direction of the beam (Mutman and Coskun, 2013; Tazabekova *et al.*, 2018).

The vibration of Euler-Bernoulli beam resting on elastic foundation has been investigated by quite a number of scholars. Eisenberger (1994) determined a general solution to vibrations of beams resting on a variable Winkler elastic foundation. Balkaya *et al*. (2009) employed the differential transform method to study the vibration analysis of beams resting on elastic foundation. The homotopy perturbation method was used by Ozturk and Coskun (2011) to analyze the vibration behaviour of beams on elastic foundation. Jiya and Shaba (2018) established the Galerkin Finite element method in conjunction with Beta time integration method to analyze a uniform Bernoulli-Euler beam subjected to a harmonic moving load on a Winkler foundation. The analysis covered the effect of acceleration of load, velocity of load and position of the load on the beam.

Ma *et al.* (2018) considered the effects of Winkler foundation mass, damping and stiffness on the nonlinear damping response of beam based on the expression of subgrade reaction obtained from the equation of motion of the Winkler foundation. The free vibration characteristics for an Euler-Bernoulli beam resting on a Winkler elastic foundation have also been studied by Tazabekova *et al.* (2018) using the He's variational iteration method. Jena *et al.* (2019) employed the differential quadrature method to study the nonlocal vibration of nanobeam resting on various types of Winkler elastic foundations such as constant, linear, parabolic, and sinusoidal types.

Oni and Awodola (2010) investigated the dynamic response under a concentrated moving mass of an elastically supported non-prismatic Bernoulli-Euler beam resting on an elastic foundation with stiffness variation. The technique used for the solution was based on the Generalized Galerkin's method and the Struble's asymptotic technique. It was found that the critical speed for the moving mass problem is reached earlier than that for the moving force problem for the illustrative examples considered. Oni and Olomofe (2011) also used the generalized Galerkin's method coupled with Struble's asymptotic technique, integral transform method and the application of the Fresnel functions to study the vibration of a non-prismatic beam resting on elastic subgrade and under the actions of accelerating masses. The dynamic behaviour of a finite uniform Rayleigh beam subjected to travelling distributed loads was studied by Andi *et al.*, (2014). It was shown that the response amplitude of the system decreases as the foundation modulus and rotatory inertia correction factor increase. It was also observed that the critical speed for the system traversed by a distributed force is greater than the one traversed by a moving distributed mass for the same natural frequency.

This paper focused on the free vibration analysis of tapered beam resting on a Winkler elastic foundation. The Rayleigh beam theory is used to model the beam and the Winkler model is considered for the elastic foundation. The effect of rotatory inertia on the natural frequencies and mode shapes of the beam is critically investigated using a semi-analytical method known as differential transform method.

2. Materials and Methods

2.1 Problem Formulation and Methods

The governing equation of motion for a prestressed non-uniform Rayleigh beam of finite length, resting on Winkler foundation as shown in figure 1 can be written as:

$$\frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2 D(x,t)}{\partial x^2}\right]+\rho(x)A(x)\frac{\partial^2 D(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[N(x)\frac{\partial D(x,t)}{\partial x}\right]$$
$$-\frac{\partial}{\partial x}\left[\rho(x)I(x)\frac{\partial^3 D(x,t)}{\partial x\partial t^2}\right]+K(x)D(x,t)=F(x,t), \quad x\in(0,l), \tag{1}$$

where $D(x,t)$ represents the dynamic response of the beam, $E(x)$ is the variable Young's modulus, $I(x)$ is the variable moment of inertia, $\rho(x)A(x)$ is the variable mass per unit length of the beam, $K(x)$ is the variable Winkler's foundation stiffness. $N(x)$ and $F(x,t)$ are arbitrary variable axial tensile and transverse excitation forces. If an axial end force N_0 is applied to the beam, then $N(x)=N_0$. However, if the distributed axial forces $g(x)$ is applied to the beam, then

$$N(x)=\int_x^l g(\eta)d\eta,$$

$\rho(x)$ is the variable density of the beam, $A(x)$ is the variable cross-section area of the beam, x is the spatial length coordinate and t is the time.

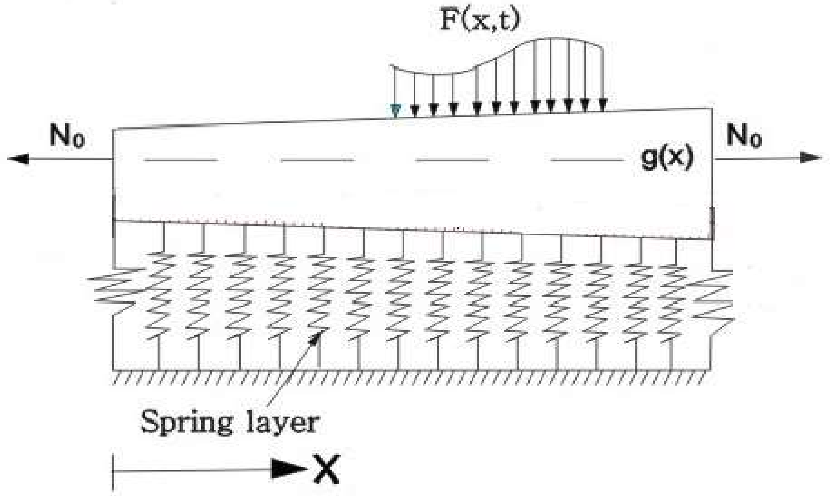

Figure 1: Model of non-uniform beam structure resting on Winkler foundation.

The initial conditions are:

$$D(x,0) = D_0(x) \quad \text{and} \quad \frac{\partial D(x,0)}{\partial t} = \dot{D}_0(x). \tag{2}$$

The relevant boundary conditions are:

Simply supported-beam:

$$D(x,t) = \frac{\partial^2 D(x,t)}{\partial x^2} = 0, \text{at} \quad x = 0,\ l. \tag{3}$$

Clamped-clamped:

$$D(x,t) = \frac{\partial D(x,t)}{\partial x} = 0, \ \text{at} \quad x = 0,\ l. \tag{4}$$

For natural vibration, $F(x,t) = 0$ and the form of ensure response is

$$D(x,t) = Y(x)e^{i\omega t} \tag{5}$$

where $Y(x)$ is the amplitude of vibration of the beam ω is the angular frequency.

Substituting equation (5) into equation (1) gives

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2Y(x)}{dx^2}\right]-\rho(x)A(x)\omega^2Y(x)-\frac{d}{dx}\left[N(x)\frac{dY(x)}{dx}\right]$$

$$+\frac{d}{dx}\left[\rho(x)I(x)\omega^2\frac{dY(x)}{dx}\right]+K(x)Y(x)=0, \quad x\in(0,l). \tag{6}$$

In Winkler modeling, the elastic foundation is represented by a set of linear springs and is assumed to vary linearly, parabolically or even constantly throughout the length of the beam Kacar *et al*. (2011). The variation of elastic coefficient of Winkler foundation is given below:

Constant:

$$k(x)=k_0, \tag{7}$$

Linear:

$$k(x)=k_0(1-\mu x), \qquad 0\le\mu\le 1. \tag{8}$$

Parabolic:

$$k(x)=k_0(1-\beta x^2), \qquad 0\le\beta\le 1. \tag{9}$$

Also, using equation (5), the boundary conditions in equations (3) and (4) are expressed as follows:

Simply supported-simply supported:

$$Y(x)=\frac{d^2Y(x)}{dx^2}=0, \text{ at } \quad x=0,\, l. \tag{10}$$

Clamped-clamped:

$$Y(x)=\frac{dY(x)}{dx}=0, \text{ at } \quad x=0,\, l. \tag{11}$$

The following dimensionless parameters are used:

$$\left.\begin{aligned}
&\xi = \frac{x}{l}, && \Lambda^2 = \frac{\rho(0)A(0)\omega^2 l^4}{E(0)I(0)},\\
&y(\xi) = \frac{Y(x)}{l}, && \psi_K = \left.\frac{h(\xi)}{c(\xi)}\right|_{\xi=1},\\
&c(\xi) = \frac{E(x)I(x)}{E(0)I(0)}, && \alpha = \frac{\rho(0)A(0)\Omega^2 l^4}{E(0)I(0)},\\
&n(\xi) = -\frac{N(x)l^2}{E(0)I(0)}, && k(\xi) = \frac{K(x)l^4}{E(0)I(0)},\\
&b(\xi) = \frac{\rho(x)A(x)}{\rho(0)A(0)}, && \tau_K = \left.\frac{c'(\xi)}{c(\xi)}\right|_{\xi=1},\\
&h(\xi) = \frac{\rho(x)I(x)}{\rho(0)I(0)}, && \eta_K = \frac{N(l)l^2}{E(l)I(l)},\\
&\text{and} && \\
&\gamma = \frac{1}{l}\sqrt{\frac{I(0)}{A(0)}}. &&
\end{aligned}\right\} \qquad (12)$$

In view of equation (12), the governing differential equation (6) and the boundary conditions given in equations (10) and (11) are written in the following dimensionless forms:

$$c(\xi)\frac{d^4y(\xi)}{d\xi^4} + 2\frac{dc(\xi)}{d\xi}\cdot\frac{d^3y(\xi)}{d\xi^3} + \frac{d^2c(\xi)}{d\xi^2}\cdot\frac{d^2y(\xi)}{d\xi^2} + n(\xi)\frac{d^2y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}$$

$$+\Lambda^2\gamma^2\left[h(\xi)\frac{d^2y(\xi)}{d\xi^2} + \frac{dh(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}\right] + \left[k(\xi) - \Lambda^2 b(\xi)\right]y(\xi) = 0,$$

(13)

Simply supported:

$$y(\xi) = \frac{d^2y(\xi)}{d\xi^2} = 0, \quad at \quad \xi = 0,1$$

(14)

Clamped-clamped:

$$y(\xi) = \frac{dy(\xi)}{d\xi} = 0, \quad at \quad \xi = 0,1 \qquad (15)$$

The dimensionless variation of elastic coefficient of Winkler foundation are given as follows:

Constant: $k(\xi) = k_0,$ (16)

Linear: $k(\xi) = k_0(1-\mu\xi)$ (17)

Parabolic: $k(\xi) = k_0(1-\beta\xi^2)$ (18)

where k_0, β and μ are constant values. Thus we have three cases.

Case 1: For constant elastic coefficient of Winkler foundation, defining $\lambda = \Lambda^2$, equation (16) is substituted into equation (13) and the differential equation takes the form

$$c(\xi)\frac{d^4y(\xi)}{d\xi^4}+2\frac{dc(\xi)}{d\xi}\cdot\frac{d^3y(\xi)}{d\xi^3}+\frac{d^2c(\xi)}{d\xi^2}\cdot\frac{d^2y(\xi)}{d\xi^2}+n(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dn(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}$$
$$+\gamma^2\lambda\left[h(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dh(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}\right]+[k_0-\lambda b(\xi)]y(\xi)=0, \tag{19}$$

Case 2: For linear elastic coefficient of Winkler foundation, setting $\lambda = \Lambda^2$, equation (17) is substituted into equation (13) and the resulting differential equation is

$$c(\xi)\frac{d^4y(\xi)}{d\xi^4}+2\frac{dc(\xi)}{d\xi}\cdot\frac{d^3y(\xi)}{d\xi^3}+\frac{d^2c(\xi)}{d\xi^2}\cdot\frac{d^2y(\xi)}{d\xi^2}+n(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dn(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}$$

$$+\gamma^2\lambda\left[h(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dh(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}\right]+k_0y(\xi)-k_0\mu\xi y(\xi)-\lambda b(\xi)y(\xi)=0, \tag{20}$$

Case 3: For parabolic variation, setting $\lambda = \Lambda^2$, equation (18) is substituted into equation (13) and the differential equation takes the form

$$c(\xi)\frac{d^4y(\xi)}{d\xi^4}+2\frac{dc(\xi)}{d\xi}\cdot\frac{d^3y(\xi)}{d\xi^3}+\frac{d^2c(\xi)}{d\xi^2}\cdot\frac{d^2y(\xi)}{d\xi^2}+n(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dn(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}$$

$$+\gamma^2\lambda\left[h(\xi)\frac{d^2y(\xi)}{d\xi^2}+\frac{dh(\xi)}{d\xi}\cdot\frac{dy(\xi)}{d\xi}\right]+k_0y(\xi)-k_0\beta\xi^2y(\xi)-\lambda b(\xi)y(\xi)=0, \tag{21}$$

Hence, the three governing differential equations considered in this paper are those in equations (19) to (21).

2.2 Method of Solution

The DTM is a transformation method based on the Taylor series expansion and is useful to obtain analytical solutions of differential equations. This method was proposed by Zhou (1986) for solving both linear and nonlinear initial value problems of electrical circuits. In this technique, certain transformation rules are applied to the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations. The solution of these algebraic equations gives the desired solution of the problem. This method gives an analytic solution in the form of a polynomial. Application of DTM leads to accurate results with fast convergence rate and small computational effort.

Basic definitions and operations of differential transform method are introduced as follows:

A function $q(t)$, analytical in domain D, can be represented by a power series around any arbitrary point in this domain. The differential transform of a function $q(t)$ is defined as:

$$\bar{Q}(k)=\frac{1}{k!}\left[\frac{d^k q(t)}{dt^k}\right]_{t=0}. \tag{22}$$

In equation (22), $q(t)$ is the original function and $\bar{Q}(k)$ is the transformed function.

The inverse differential transform of $\bar{Q}(k)$ is defined as

$$q(t)=\sum_{k=0}^{\infty}\bar{Q}(k)t^k. \tag{23}$$

Combining equations (22) and (23), this gives

$$q(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k q(t)}{dt^k}\right]_{t=0}, \tag{24}$$

which is the Taylor series of $q(t)$ at $t=0$. Equation (24) implies that the concept of differential transformation is derived from the Taylor series expansion. In practical applications, the function $q(t)$ is expressed by a finite series and equation (24) is written as

$$q(t)=\sum_{k=0}^{m}\bar{Q}(k)t^k, \tag{25}$$

which implies that

$$q(t)=\sum_{k=m+1}^{\infty}\bar{Q}(k)t^k \tag{26}$$

is negligibly small.

In this study, the value of m depends on the convergence of the natural frequencies. Table 1 contains some relevant basic operations for DTM.

Table 1: The Fundamental Operations for Differential Transform Method.

Original Function	Transformed Function
$t(x)=u(x)\pm v(x)$	$\overline{T}(k)=\overline{U}(k)\pm\overline{V}(k)$
$t(x)=\lambda u(x)$	$\overline{T}(k)=\lambda\overline{U}(k),\ \lambda$ is a constant
$t(x)=x^r$	$\overline{T}(k)=\delta(k-r)=\begin{cases}1 \text{ if } k=r\\ 0 \text{ if } k\neq r\end{cases}$
$t(x)=\dfrac{du(x)}{dx}$	$\overline{T}(k)=(k+1)\overline{U}(k+1)$
$t(x)=\dfrac{d^r u(x)}{dx^r}$	$\overline{T}(k)=(k+1)(k+2)\cdots(k+r)\overline{U}(k+r)$
$t(x)=u(x)v(x)$	$\overline{T}(k)=\sum_{r=0}^{k}\overline{U}(r)\overline{V}(k-r)=\sum_{r=0}^{k}\overline{U}(k-r)\overline{V}(r)$
$t(x)=u(x)\dfrac{dv(x)}{dx}$	$\overline{T}(k)=\sum_{r=0}^{k}\overline{U}(r)(k-r+1)\overline{V}(k-r+1)$
$t(x)=u(x)\dfrac{d^2v(x)}{dx^2}$	$\overline{T}(k)=\sum_{r=0}^{k}\overline{U}(r)(k-r+1)(k-r+2)\overline{V}(k-r+2)$

Table 2: Theorems of differential transform method for boundary conditions.

x=0		***x=l***	
Original B.C	**Transformed B.C**	**Original B.C**	**Transformed B.C**
$y(0)=0$	$Y(0)=0$	$y(1)=0$	$\sum_{k=0}^{\infty}Y(k)=0$
$\dfrac{dy(0)}{dx}=0$	$Y(1)=0$	$\dfrac{dy(1)}{dx}=0$	$\sum_{k=0}^{\infty}kY(k)=0$
$\dfrac{d^2y(0)}{dx^2}=0$	$Y(2)=0$	$\dfrac{dy^2(1)}{dx^2}=0$	$\sum_{k=0}^{\infty}k(k-1)Y(k)=0$
$\dfrac{d^3y(0)}{dx^3}=0$	$Y(3)=0$	$\dfrac{dy^2(1)}{dx^2}=0$	$\sum_{k=0}^{\infty}k(k-1)(k-2)Y(k)=0$

2.3 Application of Differential Transform Method to the Problem

Taking the differential transform of equations (19) to (21), the following recursive algebraic equations were obtained for constant, linear and parabolic variations respectively:

$$\sum_{r=0}^{k}\bar{C}(k-r)(r+1)(r+2)(r+3)(r+4)\bar{Y}(r+4)+2\sum_{r=0}^{k}(k-r+1)\bar{C}(k-r+1)(r+1)(r+2)$$
$$\times(r+3)\bar{Y}(r+3)+\sum_{r=0}^{k}(k-r+1)(k-r+2)\bar{C}(k-r+2)(r+1)(r+2)\bar{Y}(r+2)$$
$$+\sum_{r=0}^{k}\bar{N}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\sum_{r=0}^{k}(k-r+1)\bar{N}(k-r+1)(r+1)\bar{Y}(r+1)$$
$$+\lambda\gamma^2\sum_{r=0}^{k}\bar{H}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\lambda\gamma^2\sum_{r=0}^{k}(k-r+1)\bar{H}(k-r+1)(r+1)\bar{Y}(r+1)$$
$$+k_0\bar{Y}(k)=\sum_{r=0}^{k}\lambda\bar{B}(k-r)\bar{Y}(r),$$

(27)

$$\sum_{r=0}^{k}\bar{C}(k-r)(r+1)(r+2)(r+3)(r+4)\bar{Y}(r+4)+2\sum_{r=0}^{k}(k-r+1)\bar{C}(k-r+1)(r+1)(r+2)$$
$$\times(r+3)\bar{Y}(r+3)+\sum_{r=0}^{k}(k-r+1)(k-r+2)\bar{C}(k-r+2)(r+1)(r+2)\bar{Y}(r+2)$$
$$+\sum_{r=0}^{k}\bar{N}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\sum_{r=0}^{k}(k-r+1)\bar{N}(k-r+1)(r+1)\bar{Y}(r+1)$$
$$+\lambda\gamma^2\sum_{r=0}^{k}\bar{H}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\lambda\gamma^2\sum_{r=0}^{k}(k-r+1)\bar{H}(k-r+1)(r+1)\bar{Y}(r+1)$$
$$+k_0\bar{Y}(k)-\sum_{r=0}^{k}k_0\mu\delta(k-r-1)\bar{Y}(r)=\sum_{r=0}^{k}\lambda\bar{B}(k-r)\bar{Y}(r), \tag{28}$$

$$\sum_{r=0}^{k}\bar{C}(k-r)(r+1)(r+2)(r+3)(r+4)\bar{Y}(r+4)+2\sum_{r=0}^{k}(k-r+1)\bar{C}(k-r+1)(r+1)(r+2)$$
$$\times(r+3)\bar{Y}(r+3)+\sum_{r=0}^{k}(k-r+1)(k-r+2)\bar{C}(k-r+2)(r+1)(r+2)\bar{Y}(r+2)$$
$$+\sum_{r=0}^{k}\bar{N}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\sum_{r=0}^{k}(k-r+1)\bar{N}(k-r+1)(r+1)\bar{Y}(r+1)$$

$$+\lambda\gamma^2\sum_{r=0}^{k}\bar{H}(k-r)(r+1)(r+2)\bar{Y}(r+2)+\lambda\gamma^2\sum_{r=0}^{k}(k-r+1)\bar{H}(k-r+1)(r+1)\bar{Y}(r+1)$$

$$-\sum_{r=0}^{k}k_0\beta\delta(k-r-2)\bar{Y}(r)+k_0\bar{Y}(k)=\sum_{r=0}^{k}\lambda\bar{B}(k-r)\bar{Y}(r), \tag{29}$$

where $\bar{N}(k),\bar{B}(k),\bar{Y}(k),\bar{H}(k)$ and $\bar{C}(k)$ are the T-functions of $n(\xi),b(\xi),y(\xi),h(\xi)$ and $c(\xi)$ respectively. Equations (27) - (29) are algebraic equations which were implemented in MAPLE 18.

For the numerical example demonstrated in this study, the free vibration of a non-uniform simply-supported and clamped-clamped beams are considered.

The materials property of the beams are given as:

$$E(x)I(x)=E(0)I(0)\left(1-e\frac{x}{l}\right)^3 \tag{30}$$

$$\rho(x)A(x)=\rho(0)A(0)\left(1-e\frac{x}{l}\right) \tag{31}$$

where $E(0),I(0),\rho(0),e$ and $A(0)$ are Young's modulus, moment of inertia, mass per unit volume, taper ratio and cross section area at $x=0$ respectively.

The distributed axial force is given by

$$N(x)=\int_x^l\rho(\xi)A(\xi)\Omega^2\xi d\xi \tag{32}$$

$$=\rho(0)A(0)\Omega^2l^2\left[\frac{1}{2}-\frac{e}{3}-\frac{\xi^2}{2}+\frac{e\xi^3}{3}\right] \tag{33}$$

Also,

$$\left.\begin{aligned}c(\xi)&=(1-e\xi)^3\\ n(\xi)&=-\alpha^2\left(\frac{1}{2}-\frac{e}{3}-\frac{\xi^2}{2}+\frac{e\xi^3}{3}\right)\\ b(\xi)&=(1-e\xi)\\ h(\xi)&=(1-e\xi)^3\end{aligned}\right\} \tag{34}$$

In equation (27) to (29), the following terms were defined:

$$\left.\begin{aligned}
&\bar{C}(k)=\delta(k)-3e\delta(k-1)+3e^{2}\delta(k-2)-e^{3}\delta(k-3)\\
&\bar{N}(k)=-\left[\frac{1}{2}\delta(k)-\frac{e}{3}\delta(k)+\frac{1}{2}\delta(k-2+\frac{e}{3}\delta(k-3)\right]\alpha^{2}\\
&\bar{B}(k)=\delta(k)-e\delta(k-1)\\
&\bar{H}(k)=\delta(k)-3e\delta(k-1)+3e^{2}\delta(k-2)-e^{3}\delta(k-3)
\end{aligned}\right\} \qquad (35)$$

3. Result and Discussion

Table 3: Effects of inverse of slenderness ratio (γ), axial force (α) and constant elastic foundation (k_0) on dimensionless frequencies of a clamped- clamped beam

		$\alpha=0$			$\alpha=5$			$\alpha=10$		
	Winkler	k_0			k_0			k_0		
γ	λ	0	200	400	0	200	400	0	200	400
0.000	λ_1	16.3356	23.3035	28.6069	18.5105	24.8869	29.9208	23.6609	28.9440	33.3917
	λ_2	44.9806	47.9922	50.8305	48.1312	50.9581	53.6403	56.3792	58.8147	61.1558
	λ_3	88.1381	89.7193	91.2741	91.6159	93.1378	94.6362	101.2626	102.6398	104.0051
0.010	λ_1	16.3294	23.2947	28.5960	18.5031	24.8772	29.9092	23.6509	28.9323	33.3785
	λ_2	44.9232	47.9309	50.7656	48.0695	50.8928	53.5717	56.3069	58.7394	61.0778
	λ_3	87.9017	89.4785	91.0294	91.3700	92.8879	94.3821	100.9913	102.3664	103.7233
0.013	λ_1	16.3246	23.2878	28.5875	18.4981	24.8706	29.9012	23.6431	28.9233	33.3683
	λ_2	44.8786	47.8833	50.7152	48.0271	50.8480	53.5245	56.2509	58.6812	61.0172
	λ_3	87.7191	89.2928	90.8402	91.2012	92.7164	94.2080	100.7805	102.1522	103.5088
0.020	λ_1	16.3109	23.2681	28.5631	18.4810	24.8483	29.8746	23.6210	28.8975	33.3392
	λ_2	44.7521	47.7483	50.5721	47.8859	50.6986	53.3674	56.0918	58.5154	60.8453
	λ_3	87.2033	88.7679	90.3064	90.6441	92.1492	93.6324	100.1879	101.5518	102.8975
0.025	λ_1	16.2970	23.2482	28.5386	18.4645	24.8266	29.8487	23.5986	28.8714	33.3098
	λ_2	44.6250	47.6126	50.4285	47.7495	50.5543	53.2156	55.9318	58.3490	60.6725
	λ_3	86.6905	88.2455	89.7749	90.1101	91.6068	93.0808	99.5991	100.9539	102.2940

Table 4: Effects of inverse of slenderness ratio (γ), axial force (α) and linear elastic foundation (k_0) on dimensionless frequencies of a clamped- clamped beam

					$\mu=0.2$					
		$\alpha=0$			$\alpha=5$			$\alpha=10$		
	Winkler	k_0			k_0			k_0		
γ	λ	0	200	400	0	200	400	0	200	400
0.000	λ_1	16.3356	22.6425	27.5352	18.5105	24.2646	28.8895	23.6609	28.4019	32.4539
	λ_2	44.9806	47.6570	50.1927	48.1312	50.6421	53.0358	56.3792	58.5398	60.6243
	λ_3	88.1381	89.5379	90.9164	91.6159	92.9630	94.2915	101.2626	102.4823	103.6885
0.010	λ_1	16.3294	22.6339	27.5247	18.5031	24.2552	28.8784	23.6509	28.3904	32.4411
	λ_2	44.9232	47.5962	50.1286	48.0695	50.5773	52.9680	56.3069	58.4649	60.5469
	λ_3	87.9017	89.2977	90.6722	91.3700	92.7138	94.0385	100.9897	102.2075	103.4109
0.013	λ_1	16.3246	22.6272	27.5165	18.4974	24.2478	28.8697	23.6431	28.3815	32.4312
	λ_2	44.8786	47.5490	50.0789	48.0217	50.5271	52.9154	56.2509	58.4068	60.4870
	λ_3	87.7191	89.1121	90.4840	91.1799	92.5207	93.8426	100.7811	101.9955	103.1948
0.020	λ_1	16.3109	22.6081	27.4933	18.4810	24.2269	28.8450	23.6210	28.3561	32.4029
	λ_2	44.7521	47.4148	49.9376	47.8859	50.3843	52.7660	56.0917	58.2419	60.3164
	λ_3	87.2033	88.5883	89.9524	90.6439	91.9766	93.2915	100.1897	101.3949	102.5894
0.025	λ_1	16.2970	16.2970	22.5889	18.4645	24.2058	28.8201	23.5986	28.3305	32.3742
	λ_2	44.6250	44.6250	47.2801	47.7495	50.2409	52.6159	55.9318	58.0761	60.1450
	λ_3	86.6905	86.6903	88.0673	90.1100	91.4354	92.7415	99.5995	100.7993	101.9853

Table 5: Effects of inverse of slenderness ratio (γ), axial force (α) and parabolic elastic foundation (k_0) on dimensionless frequencies of a clamped- clamped beam.

$\beta = 0.1$

		$\alpha = 0$			$\alpha = 5$			$\alpha = 10$		
	Winkler	k_0			k_0			k_0		
γ	λ	**0**	**200**	**400**	**0**	**200**	**400**	**0**	**200**	**400**
0.000	λ_1	16.3356	23.1109	28.2981	18.5105	24.7042	29.6211	23.6609	28.7821	33.1142
	λ_2	44.9806	47.8813	50.6193	48.1312	50.8532	53.4397	56.3792	58.7228	60.9783
	λ_3	88.1381	89.6567	91.1505	91.6159	93.0778	94.5170	101.2621	102.5857	103.8940
0.010	λ_1	16.3294	23.1021	28.2873	18.5031	24.6946	29.6097	23.6509	28.7706	33.1011
	λ_2	44.9232	47.8201	50.5546	48.0695	50.7882	53.3713	56.3070	58.6478	60.9006
	λ_3	87.9017	89.4161	90.9063	91.3700	92.8275	94.2633	100.9900	102.3099	103.6132
0.013	λ_1	16.3246	23.0952	28.2788	18.4974	24.6871	29.6008	23.6431	28.7615	33.0910
	λ_2	44.8786	47.7727	50.5045	48.0217	50.7378	53.3183	56.2509	58.5895	60.8401
	λ_3	87.7191	89.2305	90.7174	91.1798	92.6345	94.0677	100.7811	102.0998	103.4017
0.020	λ_1	16.3109	23.0757	28.2549	18.4810	24.6658	29.5755	23.6210	28.7359	33.0622
	λ_2	44.7521	47.6380	50.3620	47.8859	50.5944	53.1678	56.0918	58.4241	60.6686
	λ_3	87.2033	88.7060	90.1841	90.6439	92.0897	93.5145	100.1874	101.4986	102.7920
0.025	λ_1	16.2970	23.0561	28.2307	18.4645	24.6443	29.5499	23.5986	28.7099	33.0330
	λ_2	44.6250	47.5026	50.2189	47.7495	50.4504	53.0166	55.9318	58.2577	60.4963
	λ_3	86.6905	88.1840	89.6533	90.1100	91.5472	92.9644	99.5994	100.9022	102.1886

Table 6: Effects of inverse of slenderness ratio (γ), axial force (α) and constant elastic foundation (k_0) on dimensionless frequencies of a simply-supported beam.

		$\alpha = 0$			$\alpha = 5$			$\alpha = 10$		
	Winkler	k_0			k_0			k_0		
γ	λ	**0**	**200**	**400**	**0**	**200**	**400**	**0**	**200**	**400**
0.000	λ_1	7.1215	17.9419	24.2970	10.3423	19.5114	25.5351	16.3263	23.3397	28.6578
	λ_2	28.9518	33.4930	37.5137	32.7157	36.7830	40.4632	41.9025	45.1425	48.1757
	λ_3	64.9788	67.1234	69.2053	68.9006	70.9236	72.8937	79.4419	81.1982	82.9198
0.010	λ_1	7.1192	17.9357	24.2882	10.3382	19.5044	25.5259	16.3192	23.3310	28.6475
	λ_2	28.9194	33.4558	37.4723	32.6783	36.7413	40.4176	41.8535	45.0901	48.1202
	λ_3	64.8206	66.9601	69.0370	68.7320	70.7503	72.7158	79.2464	80.9989	82.7162
0.013	λ_1	7.1173	17.9309	24.2814	10.3350	19.4990	25.5189	16.3137	23.3241	28.6394
	λ_2	28.8943	33.4269	37.4402	32.6493	36.7090	40.3823	41.8155	45.0495	48.0772
	λ_3	64.6983	66.8339	68.9070	68.6019	70.6164	72.5783	79.0953	80.8444	82.5589
0.020	λ_1	7.1121	17.9171	24.2619	10.3258	19.4837	25.4986	16.2981	23.3047	28.6164
	λ_2	28.8229	33.3446	37.3488	32.5669	36.6170	40.2819	41.7075	44.9340	47.9547
	λ_3	64.3527	66.4772	68.5396	68.2339	70.2380	72.1898	78.6682	80.4085	82.1141
0.025	λ_1	7.1068	17.9032	24.2423	10.3166	19.4682	25.4782	16.2823	23.2850	28.5932
	λ_2	28.7510	33.2621	37.2570	32.4840	36.5246	40.1810	41.5989	44.8180	47.8317
	λ_3	64.0083	66.1217	68.1735	67.8671	69.8609	71.8025	78.2428	79.9740	81.6706

Table 7: Effects of inverse of slenderness ratio (γ), axial force (α) and linear elastic foundation (k_0) on dimensionless frequencies of a simply-supported beam.

$\mu = 0.1$

		$\alpha = 0$			$\alpha = 5$			$\alpha = 10$		
	Winkler	k_0			k_0			k_0		
γ	λ	**0**	**200**	**400**	**0**	**200**	**400**	**0**	**200**	**400**
0.000	λ_1	7.1215	17.1398	23.1477	10.3423	18.7481	24.3956	16.3263	22.6642	27.5714
	λ_2	28.9518	32.9865	36.5887	32.7157	36.3280	39.6197	41.9025	44.7762	47.4802
	λ_3	64.9788	66.8742	68.7188	68.9006	70.6891	72.4347	79.4419	80.9955	82.5205
0.010	λ_1	7.1192	17.1340	23.1395	10.3382	18.7415	24.3872	16.3192	22.6556	27.5614
	λ_2	28.9194	32.9498	36.5480	32.6783	36.2867	39.5749	41.8535	44.7242	47.4254
	λ_3	64.8206	66.7115	68.5517	68.7320	70.5164	72.2578	79.2464	80.7962	82.3179
0.013	λ_1	7.1173	17.1295	23.1333	10.3350	18.7364	24.3806	16.3137	22.6490	27.5537
	λ_2	28.8943	32.9212	36.5165	32.6493	36.2548	39.5402	41.8155	44.6839	47.3829
	λ_3	64.6983	66.5857	68.4225	68.6019	70.3829	72.1212	79.0953	80.6423	82.1612
0.020	λ_1	7.1121	17.1166	23.1154	10.3258	18.7217	24.3617	16.2981	22.6300	27.5317
	λ_2	28.8229	32.8401	36.4269	32.5669	36.1638	39.4415	41.7075	44.5693	47.2620
	λ_3	64.3527	66.2302	68.0573	68.2339	70.0056	71.7348	78.6682	80.2075	81.7185
0.025	λ_1	7.1068	17.1035	23.0974	10.3166	18.7069	24.3426	16.2823	22.6108	27.5094
	λ_2	28.7510	32.7586	36.3368	32.4840	36.0723	39.3423	41.5989	44.4540	47.1405
	λ_3	64.0083	65.8759	67.6935	67.8671	69.6296	71.3497	78.2428	79.7738	81.2770

Table 8: Effects of inverse of slenderness ratio (γ), axial force (α) and parabolic elastic foundation (k_0) on dimensionless frequencies of a simply-supported beam.

$\beta = 0.1$

		$\alpha = 0$			$\alpha = 5$			$\alpha = 10$		
	Winkler	k_0			k_0			k_0		
γ	λ	**0**	**200**	**400**	**0**	**200**	**400**	**0**	**200**	**400**
0.000	λ_1	7.1215	17.7133	23.9781	10.3423	19.2880	25.2092	16.3263	23.1332	28.3310
	λ_2	28.9518	33.3152	37.1874	32.7157	36.6235	40.1661	41.9025	45.0139	47.9308
	λ_3	64.9788	67.0343	69.0310	68.9006	70.8399	72.7296	79.4419	81.1259	82.7774
0.010	λ_1	7.1192	17.7072	23.9695	10.3382	19.2811	25.2002	16.3192	23.1244	28.3208
	λ_2	28.9194	33.2781	37.1462	32.6783	36.5819	40.1209	41.8535	44.9616	47.8755
	λ_3	64.8206	66.8712	68.8632	68.7320	70.6668	72.5521	79.2464	80.9266	82.5742
0.013	λ_1	7.1173	17.7025	23.9629	10.3350	19.2758	25.1934	16.3137	23.1177	28.3129
	λ_2	28.8943	33.2493	37.1143	32.6493	36.5497	40.0857	41.8155	44.9211	47.8326
	λ_3	64.6983	66.7451	68.7335	68.6019	70.5330	72.4149	79.0953	80.7726	82.4170
0.020	λ_1	7.1121	17.6890	23.9440	10.3258	19.2607	25.1736	16.2981	23.0984	28.2902
	λ_2	28.8229	33.1674	37.0234	32.5669	36.4580	39.9859	41.7075	44.8059	47.7107
	λ_3	64.3527	66.3889	68.3668	68.2339	70.1550	72.0271	78.6682	80.3369	81.9731
0.025	λ_1	7.1068	17.6754	23.9250	10.3166	19.2454	25.1537	16.2823	23.0789	28.2673
	λ_2	28.7510	33.0852	36.9322	32.4840	36.3659	39.8855	41.5989	44.6901	47.5881
	λ_3	64.0083	66.0338	68.0015	67.8671	69.7782	71.6405	78.2428	79.9028	81.5304

Table 9: Convergence of first six dimensionless natural frequencies λ_1 to λ_6 of a clamped-clamped non-uniform Rayleigh beam for the three variations of elastic coefficient.

Constant		**Linear**		**Parabolic**	
m	**λ**	**m**	**λ**	**m**	**λ**
27	19.2374	27	19.1571	27	19.2135
28	48.3578	28	48.3248	28	48.3469
36	91.5231	38	91.5051	37	91.5167
48	148.7163	66	148.6982	68	148.6962
78	219.9693	70	219.6790	82	219.9817
82	304.5719	86	300.8067	84	298.2291

Table 10: Convergence of first six dimensionless natural frequencies λ_1 to λ_6 of a simply-supported non-uniform Rayleigh beam for the three variations of elastic coefficient.

Constant		**Linear**		**Parabolic**	
m	**λ**	**m**	**λ**	**m**	**λ**
29	11.5910	29	11.4614	29	11.5520
32	33.1061	32	33.0566	32	33.0888
32	68.9361	33	68.9123	33	68.9277
44	118.9426	37	118.9277	61	118.9375
72	182.9416	64	182.9690	74	182.9584
84	261.6563	84	260.4188	88	260.7044

Table 11: Comparison of the first three natural frequencies of a clamped-clamped Euler-Bernoulli beam for constant elastic modulus: $k_0 = 1, e = 0$.

Method	λ_1	λ_2	λ_3
DTM[Present]	9.92014	39.4911	88.8321
ADM [Coskun *et a*l.,2014]	9.92014	39.4911	88.8321
HPM [Mutman, 2013]	9.92014	39.4911	88.8321
DQEM [Chen, 2000]	9.92014	39.4911	88.8321

Table12: Comparison of the first three natural frequencies of a simply-supported Euler-Bernoulli beam for constant elastic modulus: $k_0 = 1, e = 0$.

Method	λ_1	λ_2	λ_3
DTM[Present]	22.3956	61.6809	120.908
ADM [Coskun *et a*l.,2014]	22.3956	61.6809	120.908
HPM [Mutman, 2013]	22.3956	61.6809	120.908
DQEM [Chen, 2000]	22.3956	61.6811	120.910

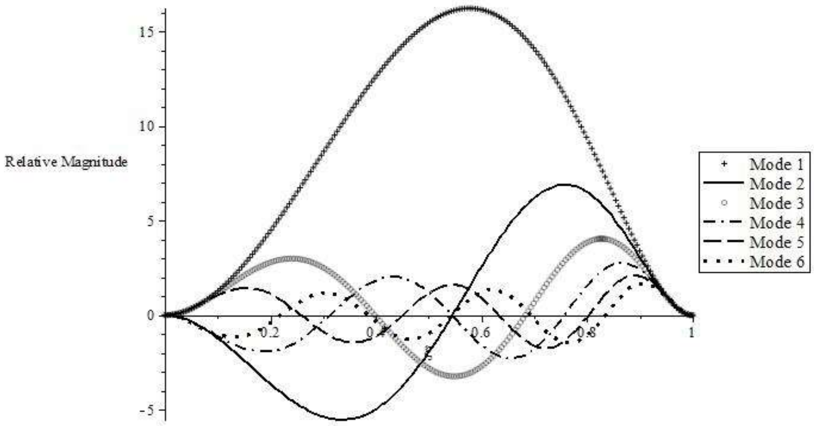

Figure 2: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for constant elastic modulus.

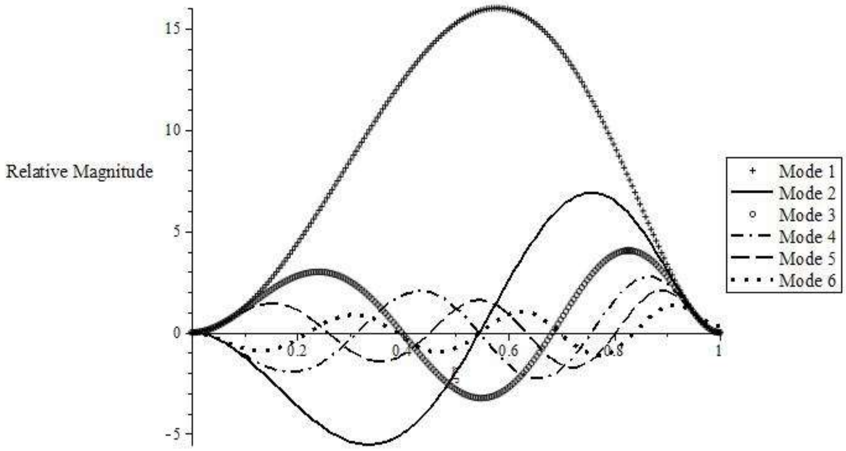

Figure 3: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for linear elastic modulus.

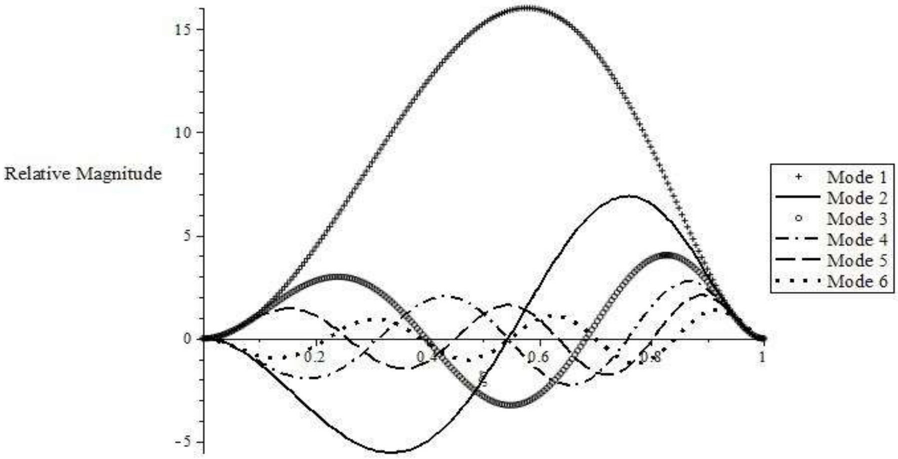

Figure 4: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for parabolic elastic modulus.

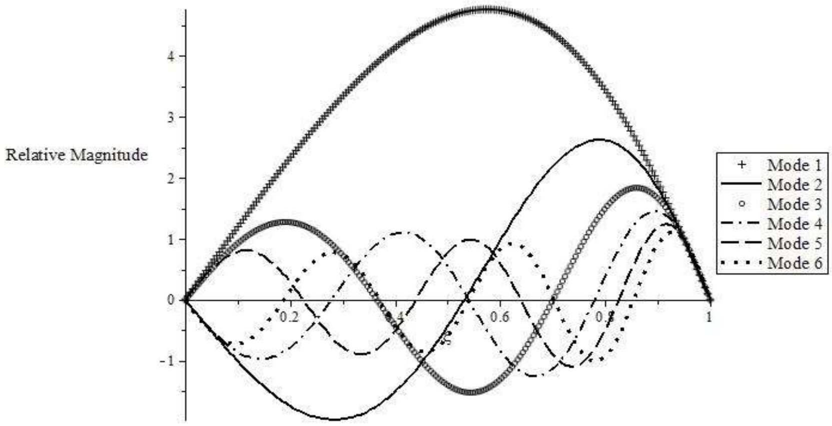

Figure 5: The first six mode shapes of a simply-supported non-uniform Rayleigh beam for constant elastic modulus.

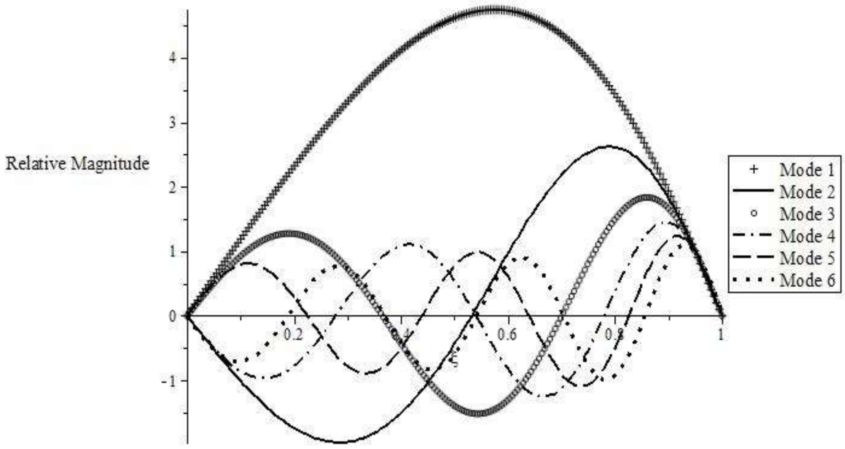

Figure 6: The first six mode shapes of a simply-supported non-uniform Rayleigh beam for linear elastic modulus.

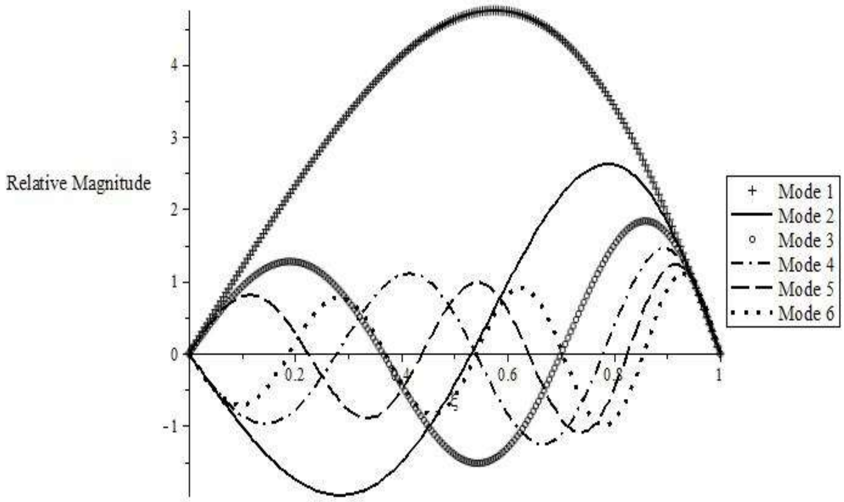

Figure 7: The first six mode shapes of a simply-supported non-uniform Rayleigh beam for parabolic elastic modulus.

Computer codes developed using MAPLE 18 were used to calculate the natural frequency and corresponding mode shape for $\gamma = 0.01$, $k_0 = 20$, $\beta = 0.1$, $\mu = 0.2, e = 0.5$ and $\alpha = 5$ using equations (27) - (29). The first six dimensionless natural frequencies λ_1 to λ_6 of a clamped-clamped and simply-supported non-uniform Rayleigh beam for the constant, linear and parabolic elastic variations were presented in Tables 9 and 10. The frequencies converged one by one without missing any one for $\epsilon = 0.0001$.

Tables 3 to 8 considered the effects of inverse of the slenderness ratio (γ), axial force (α) and elastic foundations (k_0) on the first three dimensionless natural frequencies of a clamped-clamped and simply-supported non-uniform Rayleigh beam for the constant, linear and parabolic elastic variations. It is noticed that the inverse of slenderness ratio (γ) has a reducing effect on the dimensionless natural frequencies (λ) while the increase in axial force (α) leads to an increase of the dimensionless natural frequencies. The Winkler elastic modulus (k_0) has an increasing effect on the dimensionless natural frequencies. It is observed that constant elastic modulus of the Winkler foundation has a greater effect on the dimensionless natural frequencies, followed by the parabolic elastic modulus. This is because the values of the dimensionless natural frequencies obtained are greater than that of linear variation.

To validate the method used, a comparison is made using the Euler-Bernoulli beam by setting the rotatory inertia term and the taper ratio in the governing equation of motion of a Rayleigh beam resting on the Winkler foundation to zero. In Tables 11 and 12, the DTM results for the first three dimensionless natural frequencies by methods of a clamped-clamped and simply-supported uniform beams are compared with available results in the literature. It is noticed that there is a close agreement between DTM and previously available results. In Figures 2 to 7, the mode shapes vary from one boundary condition to another.

4. Conclusion

Using DTM in this study, the closed-form series solutions of the free vibration problem of a non-uniform Rayleigh beam resting on the Winkler elastic foundation were obtained. Three cases were investigated namely, vibration problem involving constant, linear and parabolic Winkler coefficient of elastic foundations. The results obtained in this work may give information about the possible changes in vibration characteristics of engineering structures that are affected by foundation parameters and rotatory inertia. It could be applied to determine the response of beams on elastic foundations to natural phenomena such as earthquake.

References

Andi, E. A., Oni, S. T. and Ogungbamike, O. K. (2014): Dynamical Analysis of a Finite Simply Supported Uniform Rayleigh beam under travelling distributed loads. *Journal of the Nigerian Association of Mathematical Physics*. **26**, 125-136.

Balkaya, M., Kaya, M. O. and Saglamer, A. (2009): Analysis of the vibration of an elastic beam supported on elastic soil using the differential transform method. *Archive of Applied Mechanics*. **79**(2), 135 - 146.

Chen, C. N. (2000): Vibration of prismatic beam on an elastic foundation by differential quadrature element method. *Computer and Structures*. **77**, 1 - 9.

Coskun, S. B., Ozturk, B. and Mutman, U. (2014): *Adomian Decomposition Method for Vibration of Non-uniform Euler Beams on Elastic Foundation*. Proceedings of the 9th International Conference on Structural Dynamics, Porto, Portugal.

Eisenberger, M. (1994): Vibration frequencies for beams on variable one- and two-parameter elastic foundations. *Journal of Sound and Vibrations.* **176**(5), 577 - 584.

Eisenberger, M. and Clastornik, J. (1987): Vibrations and buckling of a beam on a variable Winkler elastic foundation. *Journal of Sound and Vibrations.* **115**, 233 - 241.

Ike, C. C. (2018): Fourier Sine Transform Method for the Free Vibration of Euler-Bernoulli Beam Resting on Winkler Foundation. *International Journal of Darshan Institute of Engineering Research and Emerging Technology.* **7**(1), 2018.

Jena, S. K, Chakraverty, S. and Tornabene, F. (2019): Dynamical behavior of nanobeam embedded in constant, linear, parabolic, and sinusoidal types of Winkler elastic foundation using first-Order nonlocal strain gradient model, *Materials Research Express.* **6**(8), 1 - 23.

Jiya, M. and Shaba, A. I. (2018): Analysis of a uniform Bernoulli – Euler beam on Winkler foundation subjected to harmonic moving load. *Journal of Applied Science and Environmental Management*. **22**(3), 368-372.

Kacar, A., Tan, H. T. and Kaya, M. O. (2011): Free vibration analysis of beams on variable Winkler elastic foundation by using the differential transform method. *Mathematical Computation and Applications.* **16**(3), 773 - 783.

Ma, J, Liu, F., Nie, M. and Wang J. (2018): Nonlinear free vibration of a beam on Winkler foundation with consideration of soil mass motion of finite depth. *Nonlinear Dynamics.* **92**, 429 – 441.

Mutman, U. (2013): Free Vibration Analysis of an Euler Beam of Variable Width on the Winkler Foundation Using Homotopy Perturbation Method. *Mathematical Problems in Engineering, Article ID 721294*.

Mutman, U. and Coskun, S. B. (2013): Free Vibration Analysis of Non-Uniform Euler-Bernoulli Beams on Elastic Foundation via Homotopy Perturbation Method. *World Academy of Science, Engineering and Technology International Journal of Mechanical and Mechatronics Engineering*. **7**(7), 1353 - 1358.

Oni, S. T. and Awodola, T. O. (2010): Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. *Latin American Journal of Solids and Structures.* **7**, 3-20.

Oni, S. T. and Olomofe, B. (2011) Vibration analysis of non-prismatic beam resting on elastic subgrade and under the actions of accelerating masses. *Journal of the Nigerian Mathematical Society*. **20**, 83 – 109.

Ozturk, B. and Coskun, S. B. (2011): The homotopy perturbation method for free vibration analysis of beam on elastic foundation. *Structural Engineering and Mechanics.* **37**(2), 415 - 425.

Tazabekova, A., Adair, D. Ibrayev, F. & Kim, J. (2018): *Free vibration calculations of an Euler- Bernoulli beam on an elastic foundation using He's variational iteration method*. In W. Wu and H.-S. Yu (Eds.): Proceedings of China-Europe Conference on Geotechnical Engineering, SSGG, 1022–1026.

Zhou, J. K. (1986): *Differential Transformation and its Applications for Electrical Circuits*. Huazhong University Press, Wuhan, China.