Free Vibration Analysis of Non-uniform Rayleigh Beams on Variable Winkler Elastic Foundation using Differential Transform Method

Olotu\textsuperscript{1}, O. T., Agboola\textsuperscript{2}, O. O. and Gbadeyan\textsuperscript{1}, J. A.

\textsuperscript{1} Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
\textsuperscript{2} Department of Mathematics, Covenant University, Ota, Nigeria.

Abstract

This study examines the effect of variable Winkler foundation on the natural frequencies of a prestressed non-uniform Rayleigh beam. In this work, the elastic coefficients of the foundations are assumed to vary along the length direction of the beam. A semi-analytical approach known as Differential Transform Method (DTM) is applied to the non-dimensional form of the governing equations of motion of the prestressed non-uniform Rayleigh beam and a set of recursive algebraic equations are obtained. Evaluating these derived equations and using some computer codes written and implemented in MAPLE 18, the non-dimensional frequencies and the associated mode shapes of the beam are obtained. The effects of variable Winkler foundation variations and axial force for various values of the slenderness ratio on the non-dimensional frequencies are investigated. The clamped-clamped and simply supported boundary conditions are considered to illustrate the accuracy and efficiency of this method. Finally, the results obtained are validated and are found to compare favorably well with those in the open literature.

Keyword: Free vibration, natural frequency, Winkler foundation variations and differential transform method.

1. Introduction

The problem of analyzing the vibration behaviour of beams resting on elastic foundations has a wide application in the analysis and design of the foundations of buildings, highways, railways and a host of other geotechnical structures. In fact, it is an important aspect of structural and geotechnical investigation. As a matter of fact, different beam theories namely Euler-Bernoulli, Timoshenko, Shear and Rayleigh beam theories have been used by scholars in carrying out the mathematical formulation of beam vibration problems. Amongst the models of elastic foundations that have been used in the literature is the one-parameter model known as the Winkler model (Eisenberger and Clastornik, 1987; Ike, 2018).

Corresponding Author\textsuperscript{*}: Olotu, O. T.
Email: olotu.ot@unilorin.edu.ng
Winkler foundation model, being the simplest of all the models, is the most widely used model because of its simplicity and for convenience’s sake. The Winkler model assumes that the subgrade/foundation reaction is directly proportional to the beam deflection at any point on the foundation. In other words, the soil is modelled as uniformly distributed linear elastic vertical springs, which tend to produce distributed reactions along the direction of the beam (Mutman and Coskun, 2013; Tazabekova et al., 2018).

The vibration of Euler-Bernoulli beam resting on elastic foundation has been investigated by quite a number of scholars. Eisenberger (1994) determined a general solution to vibrations of beams resting on a variable Winkler elastic foundation. Balkaya et al. (2009) employed the differential transform method to study the vibration analysis of beams resting on elastic foundation. The homotopy perturbation method was used by Ozturk and Coskun (2011) to analyze the vibration behaviour of beams on elastic foundation. Jiya and Shaba (2018) established the Galerkin Finite element method in conjunction with Beta time integration method to analyze a uniform Bernoulli-Euler beam subjected to a harmonic moving load on a Winkler foundation. The analysis covered the effect of acceleration of load, velocity of load and position of the load on the beam.

Ma et al. (2018) considered the effects of Winkler foundation mass, damping and stiffness on the nonlinear damping response of beam based on the expression of subgrade reaction obtained from the equation of motion of the Winkler foundation. The free vibration characteristics for an Euler-Bernoulli beam resting on a Winkler elastic foundation have also been studied by Tazabekova et al. (2018) using the He’s variational iteration method. Jena et al. (2019) employed the differential quadrature method to study the nonlocal vibration of nanobeam resting on various types of Winkler elastic foundations such as constant, linear, parabolic, and sinusoidal types.

Oni and Awodola (2010) investigated the dynamic response under a concentrated moving mass of an elastically supported non-prismatic Bernoulli-Euler beam resting on an elastic foundation with stiffness variation. The technique used for the solution was based on the Generalized Galerkin's method and the Struble's asymptotic technique. It was found that the critical speed for the moving mass problem is reached earlier than that for the moving force problem for the illustrative examples considered. Oni and Olomofe (2011) also used the generalized Galerkin’s method coupled with Struble’s asymptotic technique, integral transform method and the application of the Fresnel functions to study the vibration of a non-
prismatic beam resting on elastic subgrade and under the actions of accelerating masses. The
dynamic behaviour of a finite uniform Rayleigh beam subjected to travelling distributed loads
was studied by Andi et al., (2014). It was shown that the response amplitude of the system
decreases as the foundation modulus and rotatory inertia correction factor increase. It was
also observed that the critical speed for the system traversed by a distributed force is greater
than the one traversed by a moving distributed mass for the same natural frequency.

This paper focused on the free vibration analysis of tapered beam resting on a Winkler elastic
foundation. The Rayleigh beam theory is used to model the beam and the Winkler model is
considered for the elastic foundation. The effect of rotatory inertia on the natural frequencies
and mode shapes of the beam is critically investigated using a semi-analytical method known
as differential transform method.

2. Materials and Methods

2.1 Problem Formulation and Methods

The governing equation of motion for a prestressed non-uniform Rayleigh beam of finite
length, resting on Winkler foundation as shown in figure 1 can be written as:

\[
\frac{\partial^2}{\partial x^2} \left[ E(x)I(x) \frac{\partial^2 D(x,t)}{\partial x^2} \right] + \rho(x)A(x) \frac{\partial^2 D(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left[ N(x) \frac{\partial D(x,t)}{\partial x} \right] - \frac{\partial}{\partial x} \left[ \rho(x)I(x) \frac{\partial^3 D(x,t)}{\partial x \partial t^2} \right] + K(x)D(x,t) = F(x,t), \quad x \in (0,l),
\]

where \( D(x,t) \) represents the dynamic response of the beam, \( E(x) \) is the variable Young’s
modulus, \( I(x) \) is the variable moment of inertia, \( \rho(x)A(x) \) is the variable mass per unit
length of the beam, \( K(x) \) is the variable Winkler’s foundation stiffness. \( N(x) \) and \( F(x,t) \)
are arbitrary variable axial tensile and transverse excitation forces. If an axial end force \( N_0 \) is
applied to the beam, then \( N(x) = N_0 \). However, if the distributed axial forces \( g(x) \) is applied
to the beam, then

\[
N(x) = \int_0^l g(\eta)d\eta,
\]

\( \rho(x) \) is the variable density of the beam, \( A(x) \) is the variable cross-section area of the beam,
\( x \) is the spatial length coordinate and \( t \) is the time.
Figure 1: Model of non-uniform beam structure resting on Winkler foundation.

The initial conditions are:

\[ D(x,0) = D_0(x) \quad \text{and} \quad \frac{\partial D(x,0)}{\partial t} = \dot{D}_0(x). \]  

(2)

The relevant boundary conditions are:

Simply supported-beam:

\[ D(x,t) = \frac{\partial^2 D(x,t)}{\partial x^2} = 0, \text{ at } x = 0, l. \]  

(3)

Clamped-clamped:

\[ D(x,t) = \frac{\partial D(x,t)}{\partial x} = 0, \text{ at } x = 0, l. \]  

(4)

For natural vibration, \( F(x,t) = 0 \) and the form of ensure response is

\[ D(x,t) = Y(x)e^{i\omega t} \]  

(5)

where \( Y(x) \) is the amplitude of vibration of the beam \( \omega \) is the angular frequency.

Substituting equation (5) into equation (1) gives
\[
\frac{d^2}{dx^2} \left[ E(x)I(x) \frac{d^2 Y(x)}{dx^2} \right] - \rho(x)A(x)\omega^2 Y(x) - \frac{d}{dx} \left[ N(x)\frac{dY(x)}{dx} \right] + \frac{d}{dx} \left[ \rho(x)I(x)\omega^2 \frac{dY(x)}{dx} \right] + K(x)Y(x) = 0, \quad x \in (0,l).
\] (6)

In Winkler modeling, the elastic foundation is represented by a set of linear springs and is assumed to vary linearly, parabolically or even constantly throughout the length of the beam Kacar et al. (2011). The variation of elastic coefficient of Winkler foundation is given below:

**Constant:**
\[
k(x) = k_0,
\] (7)

**Linear:**
\[
k(x) = k_0(1 - \mu x), \quad 0 \leq \mu \leq 1.
\] (8)

**Parabolic:**
\[
k(x) = k_0(1 - \beta x^2), \quad 0 \leq \beta \leq 1.
\] (9)

Also, using equation (5), the boundary conditions in equations (3) and (4) are expressed as follows:

**Simply supported-simply supported:**
\[
Y(x) = \frac{d^2 Y(x)}{dx^2} = 0, \quad \text{at} \quad x = 0, \; l.
\] (10)

**Clamped-clamped:**
\[
Y(x) = \frac{dY(x)}{dx} = 0, \quad \text{at} \quad x = 0, \; l.
\] (11)
The following dimensionless parameters are used:

\[ \xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad x = \frac{E(x)I(x)}{E(0)I(0)}, \]

\[ c(\xi) = \frac{E(x)I(x)}{E(0)I(0)}, \quad b(\xi) = \frac{\rho(x)A(x)}{\rho(0)A(0)}, \quad n(\xi) = \frac{N(x)I(x)}{E(0)I(0)}, \]

\[ \beta(\xi) = \frac{\rho(x)A(x)}{\rho(0)A(0)} \]

and

\[ \gamma = \frac{1}{L} \sqrt{\frac{I(0)}{A(0)}}. \]

In view of equation (12), the governing differential equation (6) and the boundary conditions given in equations (10) and (11) are written in the following dimensionless forms:

\[ c(\xi) \frac{d^4 y(\xi)}{d\xi^4} + 2 \frac{dc(\xi)}{d\xi} \frac{d^3 y(\xi)}{d\xi^3} + \frac{d^2 c(\xi)}{d\xi^2} \frac{d^2 y(\xi)}{d\xi^2} + \frac{d^3 y(\xi)}{d\xi^3} + n(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi} \frac{dy(\xi)}{d\xi} \]

\[ + \Lambda^2 \gamma^2 \left[ h(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dh(\xi)}{d\xi} \frac{dy(\xi)}{d\xi} \right] + \left[ k(\xi) - \Lambda^2 b(\xi) \right] y(\xi) = 0, \]

(13)

Simply supported:

\[ y(\xi) = \frac{d^2 y(\xi)}{d\xi^2} = 0, \quad \text{at} \quad \xi = 0, 1 \]

(14)

Clamped-clamped:

\[ y(\xi) = \frac{dy(\xi)}{d\xi} = 0, \quad \text{at} \quad \xi = 0, 1 \]

(15)

The dimensionless variation of elastic coefficient of Winkler foundation are given as follows:

\[ k(\xi) = k_0, \]

(16)
Linear: 
\[ k(\xi) = k_0(1 - \mu \xi) \] 
(17)

Parabolic: 
\[ k(\xi) = k_0(1 - \beta \xi^2) \] 
(18)

where \( k_0, \beta \) and \( \mu \) are constant values. Thus we have three cases.

**Case 1:** For constant elastic coefficient of Winkler foundation, defining \( \lambda = \Lambda^2 \), equation (16) is substituted into equation (13) and the differential equation takes the form

\[
c(\xi) \frac{d^4 y(\xi)}{d\xi^4} + 2 \frac{d c(\xi)}{d\xi} \cdot \frac{d^3 y(\xi)}{d\xi^3} + \frac{d^2 c(\xi)}{d\xi^2} \cdot \frac{d^2 y(\xi)}{d\xi^2} + n(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \\
+ \gamma^2 \lambda \left[ h(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dh(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \right] + [k_0 - \lambda b(\xi)]y(\xi) = 0,
\]
(19)

**Case 2:** For linear elastic coefficient of Winkler foundation, setting \( \lambda = \Lambda^2 \), equation (17) is substituted into equation (13) and the resulting differential equation is

\[
c(\xi) \frac{d^4 y(\xi)}{d\xi^4} + 2 \frac{d c(\xi)}{d\xi} \cdot \frac{d^3 y(\xi)}{d\xi^3} + \frac{d^2 c(\xi)}{d\xi^2} \cdot \frac{d^2 y(\xi)}{d\xi^2} + n(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \\
+ \gamma^2 \lambda \left[ h(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dh(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \right] + k_0 y(\xi) - k_0 \mu \xi y(\xi) - \lambda b(\xi) y(\xi) = 0,
\]
(20)

**Case 3:** For parabolic variation, setting \( \lambda = \Lambda^2 \), equation (18) is substituted into equation (13) and the differential equation takes the form

\[
c(\xi) \frac{d^4 y(\xi)}{d\xi^4} + 2 \frac{d c(\xi)}{d\xi} \cdot \frac{d^3 y(\xi)}{d\xi^3} + \frac{d^2 c(\xi)}{d\xi^2} \cdot \frac{d^2 y(\xi)}{d\xi^2} + n(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \\
+ \gamma^2 \lambda \left[ h(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{dh(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \right] + k_0 y(\xi) - k_0 \beta \xi^2 y(\xi) - \lambda b(\xi) y(\xi) = 0,
\]
(21)

Hence, the three governing differential equations considered in this paper are those in equations (19) to (21).

### 2.2 Method of Solution

The DTM is a transformation method based on the Taylor series expansion and is useful to obtain analytical solutions of differential equations. This method was proposed by Zhou (1986) for solving both linear and nonlinear initial value problems of electrical circuits. In
this technique, certain transformation rules are applied to the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations. The solution of these algebraic equations gives the desired solution of the problem. This method gives an analytic solution in the form of a polynomial. Application of DTM leads to accurate results with fast convergence rate and small computational effort.

Basic definitions and operations of differential transform method are introduced as follows:

A function $q(t)$, analytical in domain $D$, can be represented by a power series around any arbitrary point in this domain. The differential transform of a function $q(t)$ is defined as:

$$
\mathcal{Q}(k) = \frac{1}{k!} \left[ \frac{d^k q(t)}{dt^k} \right]_{t=0}.
$$

In equation (22), $q(t)$ is the original function and $\mathcal{Q}(k)$ is the transformed function.

The inverse differential transform of $\mathcal{Q}(k)$ is defined as

$$
q(t) = \sum_{k=0}^{\infty} \mathcal{Q}(k)t^k.
$$

Combining equations (22) and (23), this gives

$$
q(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[ \frac{d^k q(t)}{dt^k} \right]_{t=0},
$$

which is the Taylor series of $q(t)$ at $t=0$. Equation (24) implies that the concept of differential transformation is derived from the Taylor series expansion. In practical applications, the function $q(t)$ is expressed by a finite series and equation (24) is written as

$$
q(t) = \sum_{k=0}^{m} \mathcal{Q}(k)t^k,
$$

which implies that

$$
q(t) = \sum_{k=m+1}^{\infty} \mathcal{Q}(k)t^k
$$

is negligibly small.

In this study, the value of $m$ depends on the convergence of the natural frequencies. Table 1 contains some relevant basic operations for DTM.
Table 1: The Fundamental Operations for Differential Transform Method.

<table>
<thead>
<tr>
<th>Original Function</th>
<th>Transformed Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t(x) = u(x) \pm v(x) )</td>
<td>( \tilde{T}(k) = \tilde{U}(k) \pm \tilde{V}(k) )</td>
</tr>
<tr>
<td>( t(x) = \lambda u(x) )</td>
<td>( \tilde{T}(k) = \lambda \tilde{U}(k) ), ( \lambda ) is a constant</td>
</tr>
<tr>
<td>( t(x) = x' )</td>
<td>( \tilde{T}(k) = \delta(k-r) = \begin{cases} 1 &amp; \text{if } k = r \ 0 &amp; \text{if } k \neq r \end{cases} )</td>
</tr>
<tr>
<td>( t(x) = \frac{du(x)}{dx} )</td>
<td>( \tilde{T}(k) = (k+1)\tilde{U}(k+1) )</td>
</tr>
<tr>
<td>( t(x) = \frac{d'u(x)}{dx} )</td>
<td>( \tilde{T}(k) = (k+1)(k+2) \cdots (k+r)\tilde{U}(k+r) )</td>
</tr>
<tr>
<td>( t(x) = u(x)v(x) )</td>
<td>( \tilde{T}(k) = \sum_{r=0}^{k} \tilde{U}(r)\tilde{V}(k-r) = \sum_{r=0}^{k} \tilde{U}(k-r)\tilde{V}(r) )</td>
</tr>
<tr>
<td>( t(x) = u(x)\frac{d^2v(x)}{dx^2} )</td>
<td>( \tilde{T}(k) = \sum_{r=0}^{k} \tilde{U}(r)(k-r+1)\tilde{V}(k-r+1) )</td>
</tr>
<tr>
<td>( t(x) = u(x)\frac{d^3v(x)}{dx^3} )</td>
<td>( \tilde{T}(k) = \sum_{r=0}^{k} \tilde{U}(r)(k-r+1)(k-r+2)\tilde{V}(k-r+2) )</td>
</tr>
</tbody>
</table>

Table 2: Theorems of differential transform method for boundary conditions.

<table>
<thead>
<tr>
<th>( x=0 )</th>
<th>( x=l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original B.C</td>
<td>Transformed B.C</td>
</tr>
<tr>
<td>( y(0) = 0 )</td>
<td>( Y(0) = 0 )</td>
</tr>
<tr>
<td>( \frac{dy(0)}{dx} = 0 )</td>
<td>( Y(1) = 0 )</td>
</tr>
<tr>
<td>( \frac{d^2y(0)}{dx^2} = 0 )</td>
<td>( Y(2) = 0 )</td>
</tr>
<tr>
<td>( \frac{d^3y(0)}{dx^3} = 0 )</td>
<td>( Y(3) = 0 )</td>
</tr>
</tbody>
</table>

2.3 Application of Differential Transform Method to the Problem
Taking the differential transform of equations (19) to (21), the following recursive algebraic equations were obtained for constant, linear and parabolic variations respectively:

\[
\sum_{r=0}^{k} \mathcal{C}(k-r)(r+1)(r+2)(r+3)(r+4) \mathcal{V}(r+4) + 2 \sum_{r=0}^{k} (k-r+1) \mathcal{C}(k-r+1)(r+1)(r+2) \times (r+3) \mathcal{V}(r+3) + \sum_{r=0}^{k} (k-r+1)(k-r+2) \mathcal{C}(k-r+2)(r+1)(r+2) \mathcal{V}(r+2) \\
+ \sum_{r=0}^{k} \mathcal{N}(k-r)(r+1)(r+2) \mathcal{V}(r+2) + \sum_{r=0}^{k} (k-r+1) \mathcal{N}(k-r+1)(r+1) \mathcal{V}(r+1) \\
+ \lambda \gamma^2 \sum_{r=0}^{k} \mathcal{H}(k-r)(r+1)(r+2) \mathcal{V}(r+2) + \lambda \gamma^2 \sum_{r=0}^{k} (k-r+1) \mathcal{H}(k-r+1)(r+1) \mathcal{V}(r+1) \\
+ k_0 \mathcal{V}(k) = \sum_{r=0}^{k} \lambda \mathcal{B}(k-r) \mathcal{V}(r), \tag{27}
\]

\[
\sum_{r=0}^{k} \mathcal{C}(k-r)(r+1)(r+2)(r+3)(r+4) \mathcal{V}(r+4) + 2 \sum_{r=0}^{k} (k-r+1) \mathcal{C}(k-r+1)(r+1)(r+2) \times (r+3) \mathcal{V}(r+3) + \sum_{r=0}^{k} (k-r+1)(k-r+2) \mathcal{C}(k-r+2)(r+1)(r+2) \mathcal{V}(r+2) \\
+ \sum_{r=0}^{k} \mathcal{N}(k-r)(r+1)(r+2) \mathcal{V}(r+2) + \sum_{r=0}^{k} (k-r+1) \mathcal{N}(k-r+1)(r+1) \mathcal{V}(r+1) \\
+ \lambda \gamma^2 \sum_{r=0}^{k} \mathcal{H}(k-r)(r+1)(r+2) \mathcal{V}(r+2) + \lambda \gamma^2 \sum_{r=0}^{k} (k-r+1) \mathcal{H}(k-r+1)(r+1) \mathcal{V}(r+1) \\
+ k_0 \mathcal{V}(k) - \sum_{r=0}^{k} k_0 \omega \delta(k-r-1) \mathcal{V}(r) = \sum_{r=0}^{k} \lambda \mathcal{B}(k-r) \mathcal{V}(r), \tag{28}
\]

\[
\sum_{r=0}^{k} \mathcal{C}(k-r)(r+1)(r+2)(r+3)(r+4) \mathcal{V}(r+4) + 2 \sum_{r=0}^{k} (k-r+1) \mathcal{C}(k-r+1)(r+1)(r+2) \times (r+3) \mathcal{V}(r+3) + \sum_{r=0}^{k} (k-r+1)(k-r+2) \mathcal{C}(k-r+2)(r+1)(r+2) \mathcal{V}(r+2) \\
+ \sum_{r=0}^{k} \mathcal{N}(k-r)(r+1)(r+2) \mathcal{V}(r+2) + \sum_{r=0}^{k} (k-r+1) \mathcal{N}(k-r+1)(r+1) \mathcal{V}(r+1) \\
+ k_0 \mathcal{V}(k) = \sum_{r=0}^{k} \lambda \mathcal{B}(k-r) \mathcal{V}(r),
\]
Olotu et al.  ILORIN JOURNAL OF SCIENCE

\[ + \lambda \gamma^2 \sum_{r=0}^{k} H(k-r)(r+1)(r+2) \overline{Y}(r+2) + \lambda \gamma^2 \sum_{r=0}^{k} (k-r+1) H(k-r+1)(r+1) \overline{Y}(r+1) \]

\[- \sum_{r=0}^{k} k_0 \beta \delta(k-r-2) \overline{Y}(r) + k_0 \overline{Y}(k) = \sum_{r=0}^{k} \Lambda B(k-r) \overline{Y}(r), \quad (29)\]

where \( \overline{N}(k), \overline{B}(k), \overline{Y}(k) \), and \( \overline{C}(k) \) are the T-functions of \( n(\xi), b(\xi), y(\xi), h(\xi) \) and \( c(\xi) \) respectively. Equations (27) - (29) are algebraic equations which were implemented in MAPLE 18.

For the numerical example demonstrated in this study, the free vibration of a non-uniform simply-supported and clamped-clamped beams are considered.

The materials property of the beams are given as:

\[ E(x)I(x) = E(0)I(0) \left(1 - e^{-\frac{x}{l}}\right)^3 \]

\[ \rho(x)A(x) = \rho(0)A(0) \left(1 - e^{-\frac{x}{l}}\right) \]

where \( E(0), I(0), \rho(0), e \) and \( A(0) \) are Young’s modulus, moment of inertia, mass per unit volume, taper ratio and cross section area at \( x = 0 \) respectively.

The distributed axial force is given by

\[ N(x) = \int_{\xi}^{\xi} \rho(\xi) A(\xi) \Omega^2 \xi d\xi \]

\[ = \rho(0)A(0)\Omega^2 l^2 \left[ \frac{1}{2} - \frac{e}{3} - \frac{\xi^2}{2} + \frac{e\xi^3}{3} \right] \quad (33)\]

Also,

\[
\begin{align*}
    c(\xi) & = (1 - e\xi)^3 \\
    n(\xi) & = -\alpha^2 \left( \frac{1}{2} - \frac{e}{3} - \frac{\xi^2}{2} + \frac{e\xi^3}{3} \right) \\
    b(\xi) & = (1 - e\xi) \\
    h(\xi) & = (1 - e\xi)^3
\end{align*}
\]

(34)

In equation (27) to (29), the following terms were defined:
\[
\begin{align*}
\tilde{C}(k) &= \delta(k) - 3e\delta(k-1) + 3e^2\delta(k-2) - e^3\delta(k-3) \\
\tilde{N}(k) &= -\left[\frac{1}{2}\delta(k) - \frac{e}{3}\delta(k) + \frac{1}{2}\delta(k-2) + \frac{e}{3}\delta(k-3)\right]a^2 \\
\tilde{B}(k) &= \delta(k) - e\delta(k-1) \\
\tilde{H}(k) &= \delta(k) - e^2\delta(k-1) + 3e\delta(k-2) - e^3\delta(k-3)
\end{align*}
\]

(35)

3. Result and Discussion

**Table 3:** Effects of inverse of slenderness ratio \(\gamma\), axial force \(a\) and constant elastic foundation \(k_0\) on dimensionless frequencies of a clamped-clamped beam

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>Winker</th>
<th>(k_0)</th>
<th>(\alpha = 0)</th>
<th>(k_0)</th>
<th>(\alpha = 5)</th>
<th>(k_0)</th>
<th>(\alpha = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>(\lambda_1)</td>
<td>16.3356</td>
<td>23.3035</td>
<td>28.6069</td>
<td>18.5105</td>
<td>24.8689</td>
<td>29.9208</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.9806</td>
<td>47.9922</td>
<td>50.8305</td>
<td>48.1312</td>
<td>50.9581</td>
<td>53.6403</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>88.1381</td>
<td>89.7193</td>
<td>91.2741</td>
<td>91.6159</td>
<td>93.1378</td>
<td>94.6362</td>
</tr>
<tr>
<td>0.010</td>
<td>(\lambda_1)</td>
<td>16.3294</td>
<td>23.2947</td>
<td>28.5960</td>
<td>18.5031</td>
<td>24.8772</td>
<td>29.9092</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.9232</td>
<td>47.9309</td>
<td>50.7656</td>
<td>48.0695</td>
<td>50.8928</td>
<td>53.5717</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.9017</td>
<td>89.4785</td>
<td>91.0294</td>
<td>91.3700</td>
<td>92.8879</td>
<td>94.3821</td>
</tr>
<tr>
<td>0.013</td>
<td>(\lambda_1)</td>
<td>16.3246</td>
<td>23.2878</td>
<td>28.5875</td>
<td>18.4981</td>
<td>24.8706</td>
<td>29.9012</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.8786</td>
<td>47.8833</td>
<td>50.7152</td>
<td>48.0271</td>
<td>50.8480</td>
<td>53.5245</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.7191</td>
<td>89.2928</td>
<td>90.8402</td>
<td>91.2012</td>
<td>92.7164</td>
<td>94.2080</td>
</tr>
<tr>
<td>0.020</td>
<td>(\lambda_1)</td>
<td>16.3109</td>
<td>23.2681</td>
<td>28.5631</td>
<td>18.4810</td>
<td>24.8483</td>
<td>29.8746</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.7521</td>
<td>47.7483</td>
<td>50.5721</td>
<td>47.8859</td>
<td>50.6986</td>
<td>53.3674</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.2033</td>
<td>88.7679</td>
<td>90.3064</td>
<td>90.6441</td>
<td>92.1492</td>
<td>93.6324</td>
</tr>
<tr>
<td>0.025</td>
<td>(\lambda_1)</td>
<td>16.2970</td>
<td>23.2482</td>
<td>28.5386</td>
<td>18.4645</td>
<td>24.8266</td>
<td>29.8487</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.6250</td>
<td>47.6126</td>
<td>50.4285</td>
<td>47.7495</td>
<td>50.5543</td>
<td>53.2156</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>86.6905</td>
<td>88.2455</td>
<td>89.7749</td>
<td>90.1101</td>
<td>91.6068</td>
<td>93.0808</td>
</tr>
</tbody>
</table>

**Table 4:** Effects of inverse of slenderness ratio \(\gamma\), axial force \(a\) and linear elastic foundation \(k_0\) on dimensionless frequencies of a clamped-clamped beam

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>Winker</th>
<th>(k_0)</th>
<th>(\alpha = 0)</th>
<th>(k_0)</th>
<th>(\alpha = 5)</th>
<th>(k_0)</th>
<th>(\alpha = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>(\lambda_1)</td>
<td>16.3356</td>
<td>23.6425</td>
<td>27.5352</td>
<td>18.5105</td>
<td>24.2646</td>
<td>28.8895</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.9806</td>
<td>47.6570</td>
<td>50.1927</td>
<td>48.1312</td>
<td>50.6421</td>
<td>53.0358</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>88.1381</td>
<td>89.5379</td>
<td>90.9164</td>
<td>91.6159</td>
<td>93.1378</td>
<td>94.6362</td>
</tr>
<tr>
<td>0.010</td>
<td>(\lambda_1)</td>
<td>16.3294</td>
<td>23.6339</td>
<td>27.5247</td>
<td>18.5031</td>
<td>24.2552</td>
<td>28.8784</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.9232</td>
<td>47.5962</td>
<td>50.1286</td>
<td>48.0695</td>
<td>50.5773</td>
<td>52.9680</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.9017</td>
<td>89.2977</td>
<td>90.6722</td>
<td>91.3700</td>
<td>92.7138</td>
<td>94.0385</td>
</tr>
<tr>
<td>0.013</td>
<td>(\lambda_1)</td>
<td>16.3246</td>
<td>23.6272</td>
<td>27.5165</td>
<td>18.4974</td>
<td>24.2478</td>
<td>28.8697</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.8786</td>
<td>47.5490</td>
<td>50.0789</td>
<td>48.0217</td>
<td>50.5271</td>
<td>52.9154</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.7191</td>
<td>89.1121</td>
<td>90.4840</td>
<td>91.1799</td>
<td>92.5207</td>
<td>94.8326</td>
</tr>
<tr>
<td>0.020</td>
<td>(\lambda_1)</td>
<td>16.3109</td>
<td>23.6081</td>
<td>27.4933</td>
<td>18.4810</td>
<td>24.2269</td>
<td>28.8450</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.7521</td>
<td>47.4148</td>
<td>49.9376</td>
<td>47.8859</td>
<td>50.3843</td>
<td>52.7660</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>87.2033</td>
<td>88.5883</td>
<td>90.9524</td>
<td>90.6439</td>
<td>92.9766</td>
<td>94.2915</td>
</tr>
<tr>
<td>0.025</td>
<td>(\lambda_1)</td>
<td>16.2970</td>
<td>23.5770</td>
<td>27.5889</td>
<td>18.4645</td>
<td>24.2058</td>
<td>28.8201</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2)</td>
<td>44.6250</td>
<td>47.6250</td>
<td>47.2081</td>
<td>47.7495</td>
<td>50.2409</td>
<td>52.6159</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3)</td>
<td>86.6905</td>
<td>86.6903</td>
<td>88.0673</td>
<td>90.1100</td>
<td>91.4354</td>
<td>92.7415</td>
</tr>
</tbody>
</table>
Table 5: Effects of inverse of slenderness ratio ($\gamma$), axial force ($\alpha$) and parabolic elastic foundation ($k_0$) on dimensionless frequencies of a clamped-clamped beam.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 5$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>$k_0$</td>
<td>$k_0$</td>
<td>$k_0$</td>
<td>$k_0$</td>
</tr>
<tr>
<td>0.000</td>
<td>16.3109</td>
<td>28.2878</td>
<td>18.4974</td>
<td>22.2232</td>
</tr>
<tr>
<td>0.010</td>
<td>16.3246</td>
<td>28.2788</td>
<td>18.4817</td>
<td>22.2072</td>
</tr>
<tr>
<td>0.013</td>
<td>16.3246</td>
<td>28.2788</td>
<td>18.4817</td>
<td>22.2072</td>
</tr>
<tr>
<td>0.020</td>
<td>16.2970</td>
<td>28.2307</td>
<td>18.4645</td>
<td>22.1802</td>
</tr>
<tr>
<td>0.025</td>
<td>16.2620</td>
<td>28.2037</td>
<td>18.4372</td>
<td>22.1532</td>
</tr>
</tbody>
</table>

Table 6: Effects of inverse of slenderness ratio ($\gamma$), axial force ($\alpha$) and constant elastic foundation ($k_0$) on dimensionless frequencies of a simply-supported beam.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 5$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>$k_0$</td>
<td>$k_0$</td>
<td>$k_0$</td>
</tr>
<tr>
<td>0.000</td>
<td>17.9419</td>
<td>24.2970</td>
<td>10.3423</td>
</tr>
<tr>
<td>0.010</td>
<td>17.9357</td>
<td>24.2882</td>
<td>10.3382</td>
</tr>
<tr>
<td>0.013</td>
<td>17.9309</td>
<td>24.2814</td>
<td>10.3350</td>
</tr>
<tr>
<td>0.020</td>
<td>17.9171</td>
<td>24.2619</td>
<td>10.3258</td>
</tr>
<tr>
<td>0.025</td>
<td>17.9032</td>
<td>24.2423</td>
<td>10.3166</td>
</tr>
</tbody>
</table>
Table 7: Effects of inverse of slenderness ratio ($\gamma$), axial force ($\alpha$) and linear elastic foundation ($k_0$) on dimensionless frequencies of a simply-supported beam.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$k_0$ 7.1215 17.1398 23.1477</td>
<td>$k_0$ 10.3423 17.8481 24.3956</td>
<td>$k_0$ 16.3263 22.6642 27.9714</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$k_0$ 28.9518 32.9865 36.5887</td>
<td>$k_0$ 32.7157 36.2320 39.6197</td>
<td>$k_0$ 41.9025 44.7762 47.4802</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$k_0$ 64.9788 66.8742 68.7188</td>
<td>$k_0$ 68.9006 70.6891 72.4347</td>
<td>$k_0$ 79.4419 80.9955 82.5205</td>
</tr>
</tbody>
</table>

Table 8: Effects of inverse of slenderness ratio ($\gamma$), axial force ($\alpha$) and parabolic elastic foundation ($k_0$) on dimensionless frequencies of a simply-supported beam.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$k_0$ 7.1173 17.1295 23.1333</td>
<td>$k_0$ 10.3350 17.8750 24.3800</td>
<td>$k_0$ 16.3174 22.6490 27.9537</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$k_0$ 28.8943 32.9212 36.5165</td>
<td>$k_0$ 32.6493 36.2548 39.5402</td>
<td>$k_0$ 41.8155 44.6839 47.3829</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$k_0$ 64.6983 66.5857 68.4225</td>
<td>$k_0$ 68.6019 70.3829 72.1212</td>
<td>$k_0$ 79.0953 80.6423 82.1612</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\beta = 0$</th>
<th>$\beta = 0.1$</th>
<th>$\beta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$k_0$ 7.1121 17.1136 23.1154</td>
<td>$k_0$ 10.3258 17.8217 24.3617</td>
<td>$k_0$ 16.2981 22.6500 27.9317</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$k_0$ 28.8229 32.8401 36.4269</td>
<td>$k_0$ 32.5669 36.1638 39.4415</td>
<td>$k_0$ 41.7075 44.5693 47.2620</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$k_0$ 64.3527 66.2302 68.0573</td>
<td>$k_0$ 68.2339 70.0056 71.7348</td>
<td>$k_0$ 78.6682 80.2075 81.7185</td>
</tr>
</tbody>
</table>
Table 9: Convergence of first six dimensionless natural frequencies $\lambda_1$ to $\lambda_6$ of a clamped-clamped non-uniform Rayleigh beam for the three variations of elastic coefficient.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADM [Coskun et al., 2014]</td>
<td>9.92014</td>
<td>39.4911</td>
<td>88.8321</td>
</tr>
</tbody>
</table>
Table 12: Comparison of the first three natural frequencies of a simply-supported Euler-Bernoulli beam for constant elastic modulus: $k_0 = 1, e = 0$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTM [Present]</td>
<td>22.3956</td>
<td>61.6809</td>
<td>120.908</td>
</tr>
<tr>
<td>ADM [Coskun et al., 2014]</td>
<td>22.3956</td>
<td>61.6809</td>
<td>120.908</td>
</tr>
<tr>
<td>HPM [Mutman, 2013]</td>
<td>22.3956</td>
<td>61.6809</td>
<td>120.908</td>
</tr>
<tr>
<td>DQEM [Chen, 2000]</td>
<td>22.3956</td>
<td>61.6811</td>
<td>120.910</td>
</tr>
</tbody>
</table>

Figure 2: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for constant elastic modulus.
Figure 3: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for linear elastic modulus.

Figure 4: The first six mode shapes of a clamped-clamped non-uniform Rayleigh beam for parabolic elastic modulus.

Figure 5: The first six mode shapes of a simply-supported non-uniform Rayleigh beam for constant elastic modulus.
Computer codes developed using MAPLE 18 were used to calculate the natural frequency and corresponding mode shape for $\gamma = 0.01$, $k_0 = 20$, $\beta = 0.1$, $\mu = 0.2$, $\epsilon = 0.5$ and $\alpha = 5$ using equations (27) - (29). The first six dimensionless natural frequencies $\lambda_1$ to $\lambda_6$ of a clamped-clamped and simply-supported non-uniform Rayleigh beam for the constant, linear and parabolic elastic variations were presented in Tables 9 and 10. The frequencies converged one by one without missing any one for $\epsilon = 0.0001$.

Tables 3 to 8 considered the effects of inverse of the slenderness ratio ($\gamma$), axial force ($\alpha$) and elastic foundations ($k_0$) on the first three dimensionless natural frequencies of a
clamped-clamped and simply-supported non-uniform Rayleigh beam for the constant, linear and parabolic elastic variations. It is noticed that the inverse of slenderness ratio \( (\gamma) \) has a reducing effect on the dimensionless natural frequencies \( (\lambda) \) while the increase in axial force \( (\alpha) \) leads to an increase of the dimensionless natural frequencies. The Winkler elastic modulus \( (k_o) \) has an increasing effect on the dimensionless natural frequencies. It is observed that constant elastic modulus of the Winkler foundation has a greater effect on the dimensionless natural frequencies, followed by the parabolic elastic modulus. This is because the values of the dimensionless natural frequencies obtained are greater than that of linear variation.

To validate the method used, a comparison is made using the Euler-Bernoulli beam by setting the rotatory inertia term and the taper ratio in the governing equation of motion of a Rayleigh beam resting on the Winkler foundation to zero. In Tables 11 and 12, the DTM results for the first three dimensionless natural frequencies by methods of a clamped-clamped and simply-supported uniform beams are compared with available results in the literature. It is noticed that there is a close agreement between DTM and previously available results. In Figures 2 to 7, the mode shapes vary from one boundary condition to another.

4. Conclusion

Using DTM in this study, the closed-form series solutions of the free vibration problem of a non-uniform Rayleigh beam resting on the Winkler elastic foundation were obtained. Three cases were investigated namely, vibration problem involving constant, linear and parabolic Winkler coefficient of elastic foundations. The results obtained in this work may give information about the possible changes in vibration characteristics of engineering structures that are affected by foundation parameters and rotatory inertia. It could be applied to determine the response of beams on elastic foundations to natural phenomena such as earthquake.

References


